(H.W 6) 9/30/11.

P #4.72 Two athletic teams play a series of games ; the first team to win 4 games is declared the overall winner. Suppose that one of the teams is stronger than the other and wins each game with probability 0.6 , independently of the outcomes of the other games. Find the probability, for $i = 4, 5, 6, 7$, that the stronger team wins the series in exactly i games. Compare the probability that the stronger team wins with the probability that it would win a 2-out-of-3 series.

Answer : ① $i=4$: $P(4) = \binom{4}{4}(0.6)^4(0.4)^0 = 0.1296$.

② $i=5$: In this case, since the first team to win 4 games and a series of games stop,
$P(5) = \binom{5}{4}(0.6)^4(0.4)^1 - \binom{4}{4}(0.6)^4(0.4)^1 = 0.5184$.

③ $i=6$: It is also stopped when the previous game determines the overall winner.
$P(6) = \binom{6}{4}(0.6)^4(0.4)^2 - \binom{5}{4}(0.6)^4(0.4)^2 = 0.2074$

④ $i=7$: for the same reason above,
$P(7) = \binom{7}{4}(0.6)^4(0.4)^3 - \binom{6}{4}(0.6)^4(0.4)^3 = 0.1659$.

⑤ In the case of winning a 2-out-of-3 series,
$P = \binom{3}{2}(0.6)^2(0.4)^1 - \binom{2}{2}(0.6)^2(0.4)^1 = 0.2880$.

P # 4.73. Suppose in Problem 72 that the two teams are evenly matched and each has probability $\frac{1}{2}$ of winning each game. Find the expected number of games played.

Answer : The two teams have evenly the overall winnings.

① $i=4$: $\binom{2}{1} \cdot P_1(4) = 2 \cdot \binom{4}{4}(0.5)^4 = 0.125$

② $i=5$: $\binom{2}{1} \cdot P_1(5) = 2 \cdot \left\{\binom{5}{4} - \binom{4}{4}\right\}(0.5)^5 = 0.25$

③ $i=6$: $\binom{2}{1}P_1(6) = 2\cdot\left\{\binom{6}{4}-\binom{5}{4}\right\}(0.5)^6 = 0.3125$

④ $i=7$: $\binom{2}{1}P_1(7) = 2\cdot\left\{\binom{7}{4}-\binom{6}{4}\right\}(0.5)^7 = 0.3125$

$\therefore E[X] = 4(0.125) + 5(0.25) + 6(0.3125) + 7(0.3125) = 5.8125$

P. # 4.75. A fair coin is continually flipped until heads appears for the 10-th time. Let X denote the number of tails that occur. Compute the probability mass function of X.

Answer: Let X be the number of tails.
Before heads appears for 10 th time, all other outcomes are tails. Thus, the experiment has $(10+x)$ trials. By the negative binomial Random Variable, the product mass function of X is
$P\{X=x\} = P\{Y=10+x\}$ where Y is the total number of trials.

$$P\{Y=10+x\} = \binom{(10+x)-1}{10-1}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{(10+x)-10}$$

$$= \binom{9+x}{9}\left(\frac{1}{2}\right)^{10+x} = P\{X=x\}.$$

T.EX # 36. Suppose the possible values of X are $\{x_i\}$, the possible values of Y are $\{y_j\}$, and the possible values of X+Y are $\{z_k\}$. Let A_k denote the set of all pairs of indices (i,j) such that $x_i+y_j=z_k$; that is, $A_k=\{(i,j): x_i+y_j=z_k\}$.
(a) Argue that

$$P\{X+Y=z_k\} = \sum_{(i,j)\in A_k} P\{X=x_i, Y=y_j\}.$$

Answer : $P\{X+Y=z_k\} = P\{z_k - Y = X\} = \sum_i P\{z_k - Y = x_i\}$.

$$= \sum_i P\{z_k - x_i = Y\} = \sum_i \sum_j P\{z_k - x_i = y_j\}.$$

$$= \sum_i \sum_j P\{x_i + y_j = z_k\}. \text{ (by axiom of probability)}$$

$$= \sum_{(i,j)\in A_k} P\{X=x_i, Y=y_j\}$$

because of $A_k = \{(i,j) : x_i + y_j = z_k\}$.

Q.E.D.

(b). Show that $E[X+Y] = \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}$.

Answer : $E[X+Y] = \sum_k z_k P\{X+Y=z_k\}$

$$= \sum_k z_k \sum_{(i,j)\in A_k} P\{X=x_i, Y=y_j\} \quad \text{by part (a).}$$

$$= \sum_k \sum_{(i,j)\in A_k} z_k P\{X=x_i, Y=y_j\}$$

$$= \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}$$

Q.E.D.

(c) Using the formula from part (b), argue that.

$$E[X+Y] = \sum_i \sum_j (x_i+y_j) P\{X=x_i, Y=y_j\}$$

Answer : By part (b).,

$$E[X+Y] = \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}.$$

$$= \sum_{(i,j)\in \bigcup_k A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}.$$

$$= \sum_i \sum_j (x_i+y_j) P\{X=x_i, Y=y_j\}$$

Q.E.D.

(d) Show that

① $P(X=x_i) = \sum_j P(X=x_i, Y=y_j)$

② $P(Y=y_j) = \sum_i P(X=x_i, Y=y_j)$

Answer: ① $\sum_j P(X=x_i, Y=y_j) = \sum_j P(X=x_i) \cdot P(Y=y_j \mid X=x_i)$

$= P(X=x_i) \cdot \sum_j P(Y=y_j \mid X=x_i)$

$= P(X=x_i) \cdot 1 = P(X=x_i)$

by axiom of probability, $\sum_j P(Y=y_j \mid X=x_i) = 1$.

② $\sum_i P(X=x_i, Y=y_j) = \sum_i P(Y=y_j) \cdot P(X=x_i \mid Y=y_j)$

$= P(Y=y_j) \cdot \sum_i P(X=x_i \mid Y=y_j)$

$= P(Y=y_j) \cdot 1 = P(Y=y_j)$

by axiom of probability, $\sum_i P(X=x_i \mid Y=y_j) = 1$.

(d) Prove that $E[X+Y] = E[X] + E[Y]$.

Answer: $E[X+Y] = \sum_i \sum_j (x_i + y_j) P\{X=x_i, Y=y_j\}$. by part (c).

$= \sum_i \sum_j x_i \cdot P\{X=x_i, Y=y_j\} + \sum_i \sum_j y_j \cdot P\{X=x_i, Y=y_j\}$.

$= \sum_i x_i \sum_j P\{X=x_i, Y=y_j\} + \sum_j y_j \sum_i P\{X=x_i, Y=y_j\}$

$= \sum_i x_i P(X=x_i) + \sum_j y_j P(Y=y_j)$ by part (d)

$= E[X] + E[Y]$

Q.E.D