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MATH 247 HW5

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4.28. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

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Solution: Defining the random variable X as the number of defective items in the sample, X can hold four values: 0, 1, 2, and 3. Thus the expected value of X is

$$E[X] = 0P\{X=0\} + 1P\{X=1\} + 2P\{X=2\} + 3P\{X=3\}$$

There are $\binom{20}{3}$ possible equally-likely outcomes of selecting a group of 3 items randomly from 20. There are $\binom{4}{i}$ ways of selecting a group of i items from the 4 that are defective, and $3-i$ items must be selected from the remaining 16 nondefective items to fill the rest of the sample. Thus,

$$P\{X=i\} = \frac{\binom{4}{i}\binom{16}{3-i}}{\binom{20}{3}} \cancel{=} \frac{560}{1140}$$

$$P\{X=0\} = \frac{\binom{4}{0}\binom{16}{3}}{\binom{20}{3}} = \frac{560}{1140}$$

$$P\{X=1\} = \frac{\binom{4}{1}\binom{16}{2}}{\binom{20}{3}} = \frac{(4)(120)}{1140} = \frac{480}{1140}$$

$$P\{X=2\} = \frac{\binom{4}{2}\binom{16}{1}}{\binom{20}{3}} = \frac{(6)(16)}{1140} = \frac{96}{1140}$$

$$P\{X=3\} = \frac{\binom{4}{3}\binom{16}{0}}{\binom{20}{3}} = \frac{(4)(1)}{1140} = \frac{4}{1140}$$

Substituting the above probabilities into the first equation,

$$E[X] = 0(\frac{560}{1140}) + 1(\frac{480}{1140}) + 2(\frac{96}{1140}) + 3(\frac{4}{1140})$$

$$E[X] = 0 + \frac{480}{1140} + \frac{192}{1140} + \frac{12}{1140}$$

$$E[X] = \frac{684}{1140}$$

$$E[X] = \frac{3}{5}$$

The expected number of defective items in a sample of 3 items is $\frac{3}{5}$.

4.30. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let X denote the player's winnings. Show that $E[X] = +\infty$. This problem is known as the St. Petersburg paradox.

(a) Would you be willing to pay \$1 million to play this game once?

(b) Would you be willing to pay \$1 million for each game if you could play for as long as you liked and only had to settle up when you stopped playing?

Solution: As the probability of flipping $n-1$ heads followed by a tail or the first appearance of tails being on the nth flip is $P\{X = 2^n\} = (\frac{1}{2})^{n-1}(\frac{1}{2}) = (\frac{1}{2})^n$, the expected value of X

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$$E[X] = \sum_{i=1}^{+\infty}(2^i)(\frac{1}{2})^i$$

$$E[X] = \sum_{i=1}^{+\infty} 1$$

$$E[X] = +\infty$$

(a) Though the expected winnings are ∞, the probability of winning over \$1 million is very low and the bet risky. If $X = 2^n > 1,000,000$, then $n \geq 20$. Thus, the probability that the winnings exceed the bet of \$1 million is the probability that tails first occurs on the 20th or later flip.

$$P\{n \geq 20\} = 1 - P\{n \leq 19\}$$

$$P\{n \geq 20\} = 1 - \sum_{i=1}^{19}(\frac{1}{2})^i$$

$$P\{n \geq 20\} = 1 - \frac{524287}{524288}$$

$$P\{n \geq 20\} = \frac{1}{524288}$$

Because of the (assumed)risk-averse behavior of the person offered the bet, the dollar values of possible winnings do not accurately reflect the real value of the winnings. Each additional dollar exhibits lower utility as compared to the previous one, so the real winnings may be better represented by a random variable such as $U = ln(X)$.

$$E[U] = \sum_{i=1}^{+\infty} ln(2^i)(\frac{1}{2})^i$$

$$E[U] = ln(2)\sum_{i=1}^{+\infty} i(1/2)^i$$

$$E[U] = 2ln(2)$$

$$E[U] = ln(4)$$

Then the utility of the necessary bet of \$1 million dollars, that is $ln(1000000)$ is shown to be greater than the expected value of the winnings, so the bet will not be taken.

(b) As the expected winnings of playing for m games is $E[mX] = mE[X]$ and $E[X] = +\inf$, $E[mX] = +\inf$. As the winnings increase exponentially, a single event of tails not appearing until after the 19th roll, though unlikely, can cancel out many instances of the winnings being less than the \$1 million bet, making the real winnings positive over very long sequences of games.

4.38. If $E[X] = 1$ and $Var(X) = 5$, find

(a) $E[(2+X)^2]$;

(b) $Var(4+3X)$.

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Solution: (a)

$$E[(2+X)^2] = E[X^2+4X+4]$$
$$E[(2+X)^2] = E[X^2]+E[4X]+E[4]$$

As $Var(X) = E[X^2] - E[X]^2$,

$$E[(2+X)^2] = Var(X) + (E[X])^2 + 4E[X] + 4$$
$$E[(2+X)^2] = 5 + (1)^2 + 4(1) + 4$$
$$E[(2+X)^2] = 5+1+4+4$$
$$E[(2+X)^2] = 14$$

(b) As $Var(X) = E[(X - E[X])^2]$,

$$Var(4+3X) = E[((4+3X) - E[4+3X])^2]$$
$$Var(4+3X) = E[((4+3X) - E[4] - E[3X])^2]$$
$$Var(4+3X) = E[(4+3X-4-3E[X])^2]$$
$$Var(4+3X) = E[(3X-3E[X])^2]$$
$$Var(4+3X) = E[(3(X-E[X]))^2]]$$
$$Var(4+3X) = 3^2E[(X-E[X])^2]]$$
$$Var(4+3X) = 9Var(X)$$
$$Var(4+3X) = 9(5)$$
$$Var(4+3X) = 45$$

4.41. A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times and the man is asked to predict the outcome in advance. He gets 7 out of 10 correct. What is the probability that he would have done at least this well if he had no ESP?

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Solution: Setting the random variable X to be equal to the number of coins correctly called by luck, the probability p of calling a single coin correctly is $\frac{1}{2}$ as there are two ways for the coin to be called incorrectly and two ways for it to be called correctly (one way of each for whatever the coin lands on). As guessing each flip is a Bernoulli trial, the binomial distribution function for X with $n = 10$, the total number of coin flips, and $p = \frac{1}{2}$ is

$$P\{X \geq i\} = \sum_{k=i}^{10} \binom{10}{k} (\frac{1}{2})^k (1-\frac{1}{2})^{10-k}$$

$$P\{X \leq i\} = \sum_{k=i}^{10} \binom{10}{k} (\frac{1}{2})^{10}$$

For the probability of correctly guessing 7 or more flips,

$$P\{X \geq 7\} = \sum_{k=7}^{10} \binom{10}{k} (\frac{1}{2})^{10}$$

$$P\{X \geq 7\} = P\{X=7\} + P\{X=8\} + P\{X=9\} + P\{X=10\}$$

$$P\{X \geq 7\} = \binom{10}{7}(\frac{1}{2})^{10} + \binom{10}{8}(\frac{1}{2})^{10} + \binom{10}{9}(\frac{1}{2})^{10} + \binom{10}{10}(\frac{1}{2})^{10}$$

$$P\{X \geq 7\} = (120)(\frac{1}{1024}) + (45)(\frac{1}{1024}) + (10)(\frac{1}{1024}) + (1)(\frac{1}{1024})$$
$$P\{X \geq 7\} = \frac{120}{1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$$
$$P\{X \geq 7\} = \frac{176}{1024}$$
$$P\{X \geq 7\} = \frac{11}{64}$$

The probability of correctly guessing 7 or more of 10 coin flips without ESP is $\frac{11}{64}$.

4.TE6. Let X be such that

$$P\{X = 1\} = p = 1 - P\{X = -1\}$$

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Find $c \neq 1$ such that $E[c^X] = 1$.

Solution: As the expected value of a function of a discrete random variable X equals $\sum_i f(x_i)P\{X = x_i\}$, c^X has as an expected value of

$$E[c^X] = c^1 P\{X = 1\} + c^{-1}P\{X = -1\}$$
$$E[c^X] = c(p) + (\frac{1}{c})(1 - p)$$
$$E[c^X] = cp + \frac{1-p}{c}$$

Setting this equal to 1,

$$cp + \frac{1-p}{c} = 1$$
$$pc^2 + 1 - p = c$$
$$pc^2 - c + 1 - p = 0$$

Solving this quadratic equation and disregarding the solution $c = 1$,

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$$c = \frac{1 \pm \sqrt{(-1)^2 - 4p(1-p)}}{2p}$$
$$c = \frac{1 \pm \sqrt{1 - 4p + 4p^2}}{2p}$$
$$c = \frac{1 \pm \sqrt{(2p-1)^2}}{2p}$$
$$c = \frac{1 \pm (2p-1)}{2p}$$
$$c = \frac{1 - 2p + 1}{2p}$$
$$c = \frac{2 - 2p}{2p}$$
$$c = \frac{1}{p} - 1$$

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4.TE10. Let X be a binomial random variable with parameters n and p. Show that

$$E[\frac{1}{X+1}] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

Solution: As the expected value of a function of discrete random variable X is $\sum_i f(x_i)P\{X = x_i\}$ and the probability of the binomial random variable holding a certain value is $PX = i = \binom{n}{i}p^i(1-p)^{n-i}$, the expected value of $\frac{1}{X+1}$ is

$$E[\frac{1}{X+1}] = \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}p^i(1-p)^{n-i}$$

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}\sum_{i=0}^{n} \frac{(n+1)p}{i+1}\binom{n}{i}p^i(1-p)^{n-i}$$

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}\sum_{i=0}^{n} (\frac{(n+1)p}{i+1})(\frac{n!}{i!(n-i)!})p^i(1-p)^{n-i}$$

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}\sum_{i=0}^{n} \frac{(n+1)!}{(i+1)!((n+1)-(i+1))!}p^{i+1}(1-p)^{n-i}$$

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}\sum_{i=0}^{n} \binom{n+1}{i+1}p^{i+1}(1-p)^{n-i}$$

Substituting $j = i+1$,

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}\sum_{j=1}^{n} \binom{n+1}{j}p^{j}(1-p)^{n-j+1}$$

This can be written as the sum from 0 to n minus the $j = 0$ term.

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}[-\binom{n+1}{0}p^0(1-p)^{n+1} + \sum_{j=0}^{n}\binom{n+1}{j}p^j(1-p)^{n-j+1}]$$

By the binomial theorem,

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}[-(1-p)^{n+1} + (p+(1-p))^{n+1}]$$

$$E[\frac{1}{X+1}] = \frac{1}{(n+1)p}[-(1-p)^{n+1} + (1)^{n+1}]$$

$$E[\frac{1}{X+1}] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

4.TE15. Suppose that n independent tosses of a coin having probability p of coming up heads are made. Show that the probability that an even number of heads results is $\frac{1}{2}[1+(q-p)^n]$, where $q = 1-p$. Do this by proving and then utilizing the identity

$$\sum_{i=0}^{[n/2]}\binom{n}{2i}p^{2i}q^{n-2i} = \frac{1}{2}[(p+q)^n + (q-p)^n]$$

where $[n/2]$ is the largest integer less than or equal to $n/2$. Compare this exercise with Theoretical Exercise 3.5 of Chapter 3.

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Solution: Applying binomial theorem to each of the terms of the right side of the given equation,

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n}\binom{n}{k}q^k(-p)^{n-k}]$$

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n}\binom{n}{k}q^k(-p)^{n-k}]$$

Substitution $j = n - k$ in the right sum,

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} + \sum_{j=0}^{n}\binom{n}{n-j}q^{n-j}(-p)^{j}]$$

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} + \sum_{j=0}^{n}\binom{n}{n-j}q^{n-j}(-p)^{j}]$$

As $\binom{n}{n-j} = \binom{n}{j}$,

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} + \sum_{j=0}^{n}\binom{n}{j}q^{n-j}(-p)^{j}]$$

Thus, if j is even,

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[2\sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}]$$

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}$$

If j is odd,

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = 0$$

To only consider even values of j, we substitute $2i = j$.

$$\frac{1}{2}[(p+q)^n + (q-p)^n] = \sum_{i=0}^{[n/2]}\binom{n}{2i}p^{2i} q^{n-2i}$$

As $q = 1 - p$ is given, it can substituted in to obtain

$$\frac{1}{2}[(1)^n + (q-p)^n] = \sum_{i=0}^{[n/2]}\binom{n}{2i}p^{2i} q^{n-2i}$$

$$\frac{1}{2}[1 + (q-p)^n] = \sum_{i=0}^{[n/2]}\binom{n}{2i}p^{2i} q^{n-2i}$$

The expression on the right is the probability of flipping $2i$ heads in a group of n tosses where i is an integer between 0 and $[n/2]$, meaning that an even number of heads appear in the n tosses. As the expression on the right is the probability that an even number of heads occur in n tosses of a coin, $\frac{1}{2}[1 + (q-p)^n]$ is shown to equal this probability