100 [redacted]

MA247

3.

30 Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random. What is the probability that the marble is black? What is the probability that the first box was the one selected given that the marble is white?

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Sol: Let E be the event that the marble is black,

F be the event that the first box is chosen.

$\therefore P(E) = P(EF) + P(EF^c)$

$= P(E|F)P(F) + P(E|F^c)P(F^c)$

$= \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2}$

$= \frac{7}{12}$

By Bayes' Formula, $P(F|E^c) = \frac{P(E^c|F)P(F)}{P(E^c)} = \frac{\frac{1}{2}\times\frac{1}{2}}{1-\frac{7}{12}} = \frac{3}{5}$.

$\therefore$ The probability that the marble is black is $\frac{7}{12}$.

The probability that the first box was selected given that the marble is white is $\frac{3}{5}$.

33 On rainy days, Joe is late to work with probability 0.3; on nonrainy days, he is late with probability 0.1. With probability 0.7, it will rain tomorrow.

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(a) Find the probability that Joe is early tomorrow.

Let E be the event that Joe is late tomorrow,

F be the event that it will rain tomorrow.

$\therefore P(E|F) = 0.3 \quad P(E|F^c) = 0.1 \quad P(F) = 0.7$

$P(E^c) = 1 - P(E) = 1 - (P(EF) + P(EF^c))$

$= 1 - (P(E|F)P(F) + P(E|F^c)P(F^c))$

$= 1 - (0.3 \times 0.7 + 0.1 \times 0.3)$

$= 0.76$

$\therefore$ The probability that Joe is early tomorrow is 0.76.

(b) Given that Joe was early, what is the conditional probability that it rained

$P(F|E^c) = \frac{P(FE^c)}{P(E^c)} = \frac{P(F) - P(EF)}{P(E^c)} = \frac{0.7 - 0.3\times 0.7}{0.76} = \frac{49}{76} \approx 0.64$

$\therefore$ Given that Joe was early, the conditional probability that it rained is 0.64.

62 Barbara and Dianne go target shooting. Suppose that each of Barbara's shots hits a wooden duck target with probability p_1, while each shot of Dianne's hits it with probability p_2. Suppose that they shoot simultaneously at the same target. If the wooden duck is knocked over (indicating that it was hit), what is the probability that

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(a) both shots hit the duck?

Let E be the event that Barbara hits the duck,

F be the event that Dianne hits the duck,

$$P(EF \mid E \cup F) = \frac{P(EF(E \cup F))}{P(E \cup F)} = \frac{P(EF)}{P(E \cup F)}$$

$$= \frac{P(E)P(F)}{P(E)+P(F)-P(EF)} \quad (P(EF)=P(E)P(F) \text{ since } E \text{ and } F \text{ are independent events})$$

$$= \frac{P_1 P_2}{P_1+P_2-P_1P_2}$$

$\therefore$ Given that the wooden duck is knocked over, the probability that both shots hit it is $\frac{P_1P_2}{P_1+P_2-P_1P_2}$.

(b) Barbara's shot hit the duck?

$$P(E \mid E \cup F) = \frac{P(E(E \cup F))}{P(E \cup F)} = \frac{P(E)}{P(E)+P(F)-P(E)P(F)}$$

$$= \frac{P_1}{P_1+P_2-P_1P_2}$$

$\therefore$ The probability that Barbara hits the duck is $\frac{P_1}{P_1+P_2-P_1P_2}$.

Assumption: The event that Barbara hits the target and the event that Dianne hits the target are independent.

76 Suppose that E and F are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then E will occur before F with probability $P(E)/[P(E)+P(F)]$.

Let A be the event that E occurs before F

E' be the event that the first experiment's outcome is E,

F' be the event that the first experiment's outcome is F.

$\therefore$ $P(A) = P(AE')+P(AF')+P(A(E' \cup F')^c)$ (as E', F', $(E' \cup F')^c$ are mutually exclusive)

$= P(E')+0+P(A)P((E' \cup F')^c)$ (as A and $(E' \cup F')^c$ are independent events)

$\frac{10}{10}$ $P(A) = P(E') + P(A)(1 - P(E' \cup F'))$

$= P(E') + P(A)(1 - P(E') - P(F'))$ (as $P(E \cap F) = 0$)

$\Rightarrow P(A) = \frac{P(E')}{P(E') + P(F')} = \frac{P(E)}{P(E) + P(F)}$ ($P(E) = P(E')$ $P(F) = P(F')$)

where $P(E)$ is the probability that E occurs in an experiment

$P(F)$ is the probability that F occurs in an experiment.

79 In successive rolls of a pair of fair dice, what is the probability of getting 2 sevens before 6 even numbers?

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Sol: Let E be the event that we get a seven before an even number,

F be the event that we get an even number before a seven.

By Q76, $P(E) = \frac{\frac{6}{36}}{\frac{6}{36} + \frac{18}{36}} = \frac{1}{4}$

$P(F) = \frac{\frac{18}{36}}{\frac{6}{36} + \frac{18}{36}} = \frac{3}{4}$

$\therefore$ The probability of getting 2 sevens before 6 even numbers

$= \sum_{i=0}^{5} P(2 \text{ sevens and } i \text{ even numbers})$

$= \sum_{i=0}^{5} [P(E)]^2 [P(F)]^i \binom{i+1}{1}$ (since each roll does not affect the result of the next roll; $\binom{i+1}{1}$ because the last must be a seven and ~~there are therefore~~ $\binom{i+1}{1}$ ways of lining up i even numbers and 1 seven)

$= \frac{4547}{8192}$

~~≈ 0.555~~

Chap3 Th.

6. Prove that if $E_1, E_2, \ldots, E_n$ are independent events, then

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$$P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = 1 - \prod_{i=1}^{n} [1 - P(E_i)]$$

Pf: We shall prove this by induction

Base Case: $n=1$, $P(E_1) = 1 - \prod_{i=1}^{1}(1 - P(E_1))$ True.

Inductive Step: Suppose $P(\bigcup_{i=1}^{n} E_i) = 1 - \prod_{i=1}^{n}[1 - P(E_i)]$ is true for $n=k$

Consider $n = k+1$

$$P(\bigcup_{i=1}^{k+1} E_i) = P((\bigcup_{i=1}^{k} E_i) \cup E_{k+1}) = P(\bigcup_{i=1}^{k} E_i) + P(E_{k+1}) - P((\bigcup_{i=1}^{k} E_i) \cap E_{k+1})$$

$$= P(\bigcup_{i=1}^{k} E_i) + P(E_{k+1}) - P(\bigcup_{i=1}^{k} E_i E_{k+1})$$

$$= P(\bigcup_{i=1}^{k} E_i) + P(E_{k+1}) - [\sum_{i=1}^{k} P(E_i E_{k+1}) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2} E_{k+1}) + \cdots + (-1)^{k+1} P(E_1 \cdots E_{k+1})]$$

$$= P(\bigcup_{i=1}^{k} E_i) + P(E_{k+1}) - P(E_{k+1}) P(\bigcup_{i=1}^{k} E_i)$$

$$= (1 - P(E_{k+1}))[1 - \prod_{i=1}^{k}[1 - P(E_i)]] + P(E_{k+1})$$

$= 1 - P(E_{k+1}) - \prod_{i=1}^{k+1}[1-P(E_i)] + P(E_{k+1})$

$= 1 - \prod_{i=1}^{k+1}[1-P(E_i)]$

10 Consider a collection of n individuals. Assume that each person's birthday is equally likely to be any of the 365 days of the year and also that the birthdays are independent. Let $A_{i,j}$, $i \neq j$, denote the event that person i and j have the same birthday. Show that these events are pairwise independent, but not independent. That is, show that $A_{i,j}$ and $A_{r,s}$ are independent, but the $\binom{n}{2}$ events $A_{i,j}$, $i \neq j$ are not independent.

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Pf: $P(A_{i,j}A_{r,s}) = P(A_{i,j}|A_{r,s})P(A_{r,s})$

$= \frac{1}{365} \times \frac{1}{365} \qquad (\{i,j\} \neq \{r,s\})$

$= P(A_{i,j})P(A_{r,s})$

$\therefore A_{i,j}, A_{r,s}$ are independent.

However, $P(\bigcap_{i \neq j} A_{i,j}) = (\frac{1}{365})^{n-1}$ (as $\bigcap_{i\neq j} A_{i,j}$ means all the n people have the same birthday)

$\prod_{i \neq j} P(A_{i,j}) = (\frac{1}{365})^{\binom{n}{2}}$

Since $\binom{n}{2} = n-1 \iff n = 1$ or 2, for $n \geq 3$, $\binom{n}{2} \neq n-1$

$\Rightarrow P(\bigcap_{i\neq j} A_{i,j}) \neq \prod_{i \neq j} P(A_{i,j})$

$\Rightarrow$ The $\binom{n}{2}$ events are not independent.

28 Prove or give a counterexample. If E_1 and E_2 are independent, then they are conditionally independent given F.

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Counterexample: Two coins are flipped. E_1 is the event that the first coin gets a head, E_2 is the event that the second one gets a head. F is the event that out of the two coins, we get exactly one head.

$P(E_1) = \frac{1}{2} \quad P(E_2) = \frac{1}{2} \quad P(E_1E_2) = P(E_1)P(E_2) = \frac{1}{4}$,

but $P(E_1E_2|F) = 0 \quad P(E_1|F) = \frac{P(E_1F)}{P(F)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \quad P(E_2|F) = \frac{1}{2}$

$\therefore P(E_1E_2|F) \neq P(E_1|F)P(E_2|F)$

They are not conditionally independent given F.

4.

17 Suppose that the distribution function of X is given by

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$$F(b) = \begin{cases} 0 & b<0 \\ \frac{b}{4} & 0\le b<1 \\ \frac{1}{2}+\frac{b-1}{4} & 1\le b<2 \\ \frac{11}{12} & 2\le b<3 \\ 1 & 3\le b \end{cases}$$

(a) Find $P\{X=i\}, i=1,2,3$

Sol: $P\{X=1\} = P\{X\le 1\} - P\{X<1\}$

$= \frac{1}{2}+\frac{1-1}{4} - \lim_{n\to\infty} F(1-\frac{1}{n})$

$= \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{4}$

$P\{X=2\} = P\{X\le 2\} - P\{X<2\}$

$= \frac{11}{12} - (\frac{1}{2}+\frac{2-1}{4})$

$= \frac{1}{6}$

$P\{X=3\} = P\{X\le 3\} - P\{X<3\}$

$= 1 - \frac{11}{12}$

$= \frac{1}{12}$

(b) Find $P\{\frac{1}{2}<X<\frac{3}{2}\}$.

Sol: $P\{\frac{1}{2}<X<\frac{3}{2}\} = P(X<\frac{3}{2}) - P(X\le\frac{1}{2})$

$= \frac{1}{2}+\frac{\frac{3}{2}-1}{4} - \frac{\frac{1}{2}}{4}$

$= \frac{1}{2}$

25. Two coins are to be flipped. The first coin will land on heads with probabili 0.6, the second with probability 0.7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

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(a) Find $P\{X=1\}$.

$P\{X=1\} = P\{(H,T)\} \cup P\{(T,H)\}$

$= 0.6\times(1-0.7)+(1-0.6)\times 0.7$

$= 0.46$

(b). Determine $E[X]$.

$P\{X=0\} = P\{(T,T)\} = (1-0.7)\times(1-0.6) = 0.12$

$P\{X=2\} = P\{(H,H)\} = 0.7\times 0.6 = 0.42$

$\therefore E[X] = 0\times 0.12 + 1\times 0.46 + 2\times 0.42 = 1.3$.