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Homework 3 Math 247

19. Q: Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color face up?

$\frac{10}{10}$

A: Let R be the event that both dice land on Red
Similarly, B for black, Y for Yellow, and W for white
Then, $P(\text{same color}) = P(R) + P(B) + P(Y) + P(W)$.

$P(R) = \left(\frac{2}{6}\right)\left(\frac{2}{6}\right) = \frac{4}{36}$ $\qquad$ $P(Y) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$

$P(B) = \left(\frac{2}{6}\right)\left(\frac{2}{6}\right) = \frac{4}{36}$ $\qquad$ $P(W) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$

Thus,
$$P(\text{same color}) = \frac{4}{36} + \frac{4}{36} + \frac{1}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$$

23. Q: A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than the first die?

$\frac{10}{10}$

A: If the first die is a 1, there are 5 possible outcomes for the second roll which are higher than the first (2, 3, 4, 5, 6).
Similarly, if the first roll is a 2, there are 4 possible outcomes for the second roll in which the value of the 2nd roll is higher than the value of the first (3, 4, 5, 6).
This pattern continues until the final case in which the first roll is a 6 and there are no possible values for the second roll which could be higher.
The total probability then

$$P(S>F) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{4}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{3}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{0}{6}\right)$$

$$= \frac{15}{36} = \frac{5}{12}$$

↑ these $\left(\frac{1}{6}\right)$ represent the even probabilities of outcomes on the first roll

27. An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

10/10

Let R_i be the event that the first red ball is taken on the i^{th} draw.

Then A wins if R_i, $i = 1, 3, 5, 7$ (not 9 because if no reds have been pulled in the first 7 draws, then all 3 remaining balls are red and B has the next draw)

Considering R_1, if the first ball drawn is red, the remaining 9 could be drawn in any order. Thus there are $\binom{9}{2} = 36$ possible outcomes in R_1.

Similarly for R_3, if the first two balls are black and then the third is red, the remaining 7 can be drawn in any order. Thus there are $\binom{7}{2} = 21$ outcomes in R_3.

Continuing, $|R_5| = \binom{5}{2} = 10$ $\quad |R_7| = \binom{3}{2} = 3$

Lastly, the size of the sample space is just the number of possible ways to select a sequence of 10 balls, with 3 red and 7 black. There are $\binom{10}{3} = 120$ ways to do this.

Thus, $P(A \text{ wins}) = \dfrac{|R_1 \cup R_3 \cup R_5 \cup R_7|}{|S|} = \dfrac{36+21+10+3}{120} = \dfrac{70}{120} = \dfrac{7}{12}$

32. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the i^{th} position, $1 \le i \le b+g$, is a girl?

10/10

Let G_i be the event that the person in the i^{th} position is a girl.
Let the sample space S is all the ways to order $(b+g)$ people, b boys and g girls.

Then $P(G_i) = \dfrac{|G_i|}{|S|}$.

$$|S| = \binom{b+g}{g} = \frac{(b+g)!}{b!\,g!}$$

* for the purposes of this problem, individuals of the same sex are indistinguishable

$$|G_i| = \binom{b+g-1}{g-1} = \frac{(b+g-1)!}{(g-1)!\,b!}$$

* since, if we put a girl in the i^{th} position, the other $(b+g-1)$ positions can be filled in randomly

Then,
$$P(G_i) = \frac{\left(\frac{(b+g-1)!}{(g-1)!\,b!}\right)}{\left(\frac{(b+g)!}{b!\,g!}\right)}$$

$$= \frac{(b+g-1)!\,b!\,g!}{(b+g)!\,(g-1)!\,b!} = \frac{g}{(b+g)}$$

46. How many people have to be in a room in order that the probability that at least two of them celebrate their birthday in the same month is at least $\frac{1}{2}$? Assume that all possible monthly outcomes are equally likely.

$\frac{10}{10}$

Let A_n be the event that at least two of the n people in the room have the same birth month.

Then, we want the smallest n such that $P(A_n) > \frac{1}{2}$, or equivalently

$1 - P(A_n^c) > \frac{1}{2} \rightarrow P(A_n^c) < \frac{1}{2}$

where A_n^c is the event that none of the n people in the room have the same birth month.

$$P(A_n^c) = \frac{|A_n^c|}{|S|}$$

$|S| = 12^n$ * 12 possible birth months for each of the n people

$$|A_n^c| = \underbrace{\binom{12}{n}}_{\text{number of ways to choose } n \text{ months}} \underbrace{n!}_{\text{number of ways to assign } n \text{ unique birthmonths to } n \text{ people.}} = \frac{12!}{(12-n)!} \quad 0 \le n \le 12$$

* Since there are 12 months, no more than 12 people can be in a group without at least two sharing a birth month

Then, $P(A_n^c) = \frac{12!}{(12-n)!\,12^n} < \frac{1}{2}$

by trial and error,

$n = 4$, $\frac{12!}{8!\,12^4} = \frac{11 \times 10 \times 9}{12^3} = 0.57 > \frac{1}{2}$

$n = 5$, $\frac{12!}{7!\,12^5} = \frac{11 \times 10 \times 9 \times 8}{12^4} = 0.3819 < \frac{1}{2}$.

Thus, there must be at least 5 people in a room in order that the probability that at least two of them celebrate their birthday in the same month is at least $\frac{1}{2}$.

TE 18. Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear. Argue that

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2 \quad f_0 \equiv 1, \; f_1 \equiv 2$$

$\frac{10}{10}$

(1) • The n^{th} toss can be either a head or a tail.

(2) • If the n^{th} toss comes up heads, we know that the $(n-1)^{th}$ toss must have been tails (since no successive heads appear). Further, we know that the previous $(n-2)$ must not have resulted in successive heads, or there wouldn't be a need for the $(n-1)^{th}$ and n^{th} toss. There are then f_{n-2} ways of achieving this outcome.

(3) • If the n^{th} toss is tails, then we know that there are no successive heads in the previous $(n-1)$ tosses. There are f_{n-1} ways of achieving this.

• Since $E = \{n^{th} \text{ toss} = H \cap \text{no successive heads}\}$ and $F = \{n^{th} \text{ toss } T \cap \text{no successive } H\}$ are disjoint events, $|E \cup F| = |E| + |F|$

• as shown in (2), $|E| = f_{n-2}$

• as shown in (3), $|F| = f_{n-1}$

Thus, $f_n = f_{n-1} + f_{n-2}$ □.

TE 20. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occuring?

$\frac{10}{10}$

Let $E_1, E_2, \ldots \subset S$ be ~~points~~ disjoint events in S

Also, lets consider the case where all points are equally likely,

$$P(\{E_i\}) = p \quad \text{for} \quad i = 1, 2, \ldots$$

where $0 < p \leq 1$.

Then $\sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} p \to \infty$ for $p > 0$.

However, $P(S) = 1$ so we have arrived at a contradiction.

Thus, not all points can be equally likely.

All points can, however, have a positive probability of occuring.

Consider for instance $P(E_i) = \left(\frac{1}{2}\right)^i$

Then $\forall i < \infty$ $P(E_i) > 0$ and $\sum^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2} \sum^{\infty} \left(\frac{1}{2}\right)^i = \left(\frac{1}{1-\frac{1}{2}}\right)\frac{1}{2} = 1 = P(S)$.

7. The King comes from a family of 2 children. What is the probability that the other child is his sister?

$\frac{10}{10}$

Let G be the event that the King's sibling is a girl.

Let K be the event that at least one of the 2 children is a boy

$$P(G|K) = \frac{P(GK)}{P(K)} = P(GK)$$

There are 4 possible outcomes of choosing the sex of 2 children when we include birth order: $(b,b), (b,g), (g,b), (g,g)$.

$$P(GK) = P((g,b) \cup (b,g)) = \frac{2}{4}$$

$$P(K) = \frac{3}{4}$$

$$\rightarrow P(G|K) = \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

8. A couple has 2 children. What is the probability that both are girls if the older of the two is a girl?

$\frac{10}{10}$

There are 4 possible outcomes of choosing the sex of 2 children when we account for birth order:

(b,b), (g,g), (b,g), (g,b).

Let $A = \{(g,g)\}$ be the event that both children are girls

Let $B = \{(g,b), (g,g)\}$ be the event that the older child is a girl.

Then $P(A|B) = \frac{P(AB)}{P(B)}$

$$P(AB) = P(A) = \frac{1}{4} \quad \text{since} \quad A \subset B$$

$$P(B) = \frac{2}{4}$$

$$P(A|B) = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

15\. An ectopic pregnancy is twice as likely to develop when the pregnant woman is a smoker as it is when she is a nonsmoker. if 32 percent of women of childbearing age are smokers, what percentage of women having ectopic pregnancies are smokers?

$\frac{10}{10}$

Let A be the event that the woman is a smoker.
Then A^c is the event that the woman is a nonsmoker.

Given: $P(A) = 0.32 \rightarrow P(A^c) = 0.68$

Let E be the event of an ectopic pregnancy.

Given: $P(E \mid A) = 2P(E \mid A^c)$

We want to find $P(A \mid E)$

$$P(A \mid E) = \frac{P(AE)}{P(E)} = \frac{P(E \mid A)P(A)}{P(E \mid A)P(A) + P(E \mid A^c)P(A^c)}$$

$$= \frac{P(E \mid A) \cdot 0.32}{P(E \mid A)\left[0.32 + 0.5(0.68)\right]}$$

$$= \frac{0.32}{0.32 + 0.34}$$

$$= 0.\overline{48}$$