Math 247 - Exam 3, November 30, 2011 Name: Key

Problem 1 (20 points). Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed normal random variables with mean 5 and variance 2. Let N be a Poisson random variable with parameter 6, which is independent of the X_k's. Compute $E[\Sigma_{k=1}^N X_k]$.

$$E\left[\sum_{k=1}^{N} X_k\right] = \sum_{n=1}^{\infty} E\left[\sum_{k=1}^{n} X_k \mid N=n\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} n\, E[X_1] \cdot P(N=n)$$

$$= E[X_1]\, E[N] = 5 \cdot 6 = 30$$

Problem 2 (20 points). In each of the following cases determine if the two random variables X and Y are negatively correlated, positively correlated, or have no correlation. Justify your answer.

a) Flip a fair coin 10 times. Let X be the number of heads, and let Y be the number of tails flipped before the first head appears.

b) X and Z are independent and identically distributed, and $Y = X + Z$.

c) Z_1 and Z_2 are identically distributed, $X = Z_1 + Z_2$ and $Y = Z_1 - Z_2$.

a) The larger Y is the more tails we have which means the less heads we have $\therefore$ X & Y are negatively correlated.

b) $\operatorname{cov}(X, X+Z) = \operatorname{cov}(X,X) + \operatorname{cov}(X,Z)$
$= \operatorname{Var}(X) + 0$
> 0 if X is not constant with probability 1.

c) $\operatorname{cov}(Z_1+Z_2, Z_1-Z_2)$
$= \operatorname{cov}(Z_1,Z_1) + \operatorname{cov}(Z_2,Z_1) - \operatorname{cov}(Z_1,Z_2) - \operatorname{cov}(Z_2,Z_2)$
$= \operatorname{Var}(Z_1) - \operatorname{Var}(Z_2) = 0.$

Problem 3 (20 points). Consider the experiment of repeatedly rolling a die until a 6 appears. Suppose X is the sum of the values of all the rolls, find $E[X]$.

Conditioning on the first roll X_1 we have

$$E[X] = \sum_{n=1}^{6} E[X \mid X_1 = n]\, P(X_1 = n)$$

$$= \sum_{n=1}^{5} (n + E[X]) \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \frac{1}{6}\frac{5 \cdot 6}{2} + \frac{5}{6} E[X] + 1$$

$$\therefore\ E[X] = 6\left(\frac{15}{6} + 1\right) = 21$$

Problem 4 (20 points). Consider an urn containing a large number of coins, and suppose that each of the coins has some probability p of turning up heads when it is flipped. However, this value of p varies from coin to coin. Suppose that the composition of the urn is such that if a coin is selected at random from it, then the p-value of the coin can be regarded as being the value of a random variable that is uniformly distributed over $[0, 1]$. If a coin is selected at random from the urn and flipped twice, compute the probability that both flips result in heads.

$$P(hh) = \int_0^1 P(hh \mid X \text{ is Bernoulli w/ parameter } p)\, dp$$

$$= \int_0^1 p^2\, dp = \frac{1}{3}$$

Problem 5 (20 points). Let X be and Y be independent exponential random variables with parameter λ. Compute $E[(X+Y)^{10}]$ and simplify your answer.

$$M_X(t) = M_Y(t) = \frac{\lambda}{\lambda - t}$$

$$\therefore M_{(X+Y)}(t) = \frac{\lambda^2}{(\lambda - t)^2}$$

$$\therefore M'_{(X+Y)}(t) = \frac{2\lambda^2}{(\lambda - t)^3}$$

$$\vdots$$

$$M^{(10)}_{(X+Y)}(t) = \frac{11!\, \lambda^2}{(\lambda - t)^{12}}$$

$$\therefore E[(X+Y)^{10}] = M^{(10)}_{(X+Y)}(0) = \frac{11!}{\lambda^{10}}.$$