Math 247 - Exam 1, September 28, 2011 Name: Key

Problem 1 (20 points). Julie has just finished drying her laundry which contained 5 pairs of black socks, 2 pairs of blue socks, 2 pairs of yellow socks, and no other socks. Without looking Julie grabs two socks from the dryer.

a) What is the probability that Julie grabbed a matching pair of socks.

b) Suppose you knew somehow that at least one of the socks Julie grabbed was yellow. Now what is the probability that Julie grabbed a matching pair of socks

a) Total # of Socks $= 2(5+2+2) = 18$

Total # of pairs $= \binom{18}{2}$

Total # of black pairs $= \binom{10}{2}$

" blue " $= \binom{4}{2}$

" yellow " $= \binom{4}{2}$

Probability of matching pair $= \dfrac{\binom{10}{2}+\binom{4}{2}+\binom{4}{2}}{\binom{18}{2}} = \dfrac{57}{153} = \boxed{\dfrac{19}{51}}$

b) Total # of pairs where exactly one is yellow $= \binom{4}{1}\binom{16}{1}$

total # of pairs where both are yellow $= \binom{4}{2}$

Probability of matching pair given that at least 1 is yellow

$= \dfrac{\binom{4}{2}}{\binom{4}{1}\binom{14}{1}+\binom{4}{2}} = \dfrac{6}{62} = \boxed{\dfrac{3}{31}}$

Problem 2 (20 points). A certain birth defect occurs in 12% of the population. However, if a family has n children and none of the children have the birth defect, then the probability of the next child born into the family to have the defect goes down to $(12/(n+1))\%$.

The Smith family has 4 children. What is the probability that none of the Smith children have this defect?

Let E_i be the event that the i^{th} oldest Smith child has the defect.

We know from above that $P(E_n | E_1^c E_2^c \cdots E_{n-1}^c) = \frac{.12}{n}$

By the chain rule

$$P(E_1^c E_2^c E_3^c E_4^c) = P(E_1^c) P(E_2^c | E_1^c) P(E_3^c | E_1^c E_2^c) P(E_4^c | E_1^c E_2^c E_3^c)$$

$$= (1 - .12)\left(1 - \frac{.12}{2}\right)\left(1 - \frac{.12}{3}\right)\left(1 - \frac{.12}{4}\right)$$

$$= (.88)(.94)(.96)(.97)$$

$$\approx 77\%$$

Problem 3 (20 points). Prove or give counter-examples to the following statements.

a) For any events E and F, with $P(F) \neq 0$ we have $P(F \mid E \cup F) \geq P(E \mid F)$.

b) For any events E and F, with $P(F) \neq 0$ we have $P(E \mid E \cup F) \geq P(E \mid F)$.

c) For any independent events E and F we have $P(E^c \cup F) - P(F) = P(E \cup F^c) - P(E)$.

a) This is false. Consider e.g. if $E = S$ is the whole space while $P(F) \neq 0, 1$. Then

$$P(F \mid E \cup F) = P(F) < 1$$

while $P(E \mid F) = 1$

---

b) This is true. If we write $a = P(EF^c)$ $b = P(EF)$ and $c = P(E^cF)$ Then the left hand side is

$$P(E \mid E \cup F) = \frac{P(E(E \cup F))}{P(E \cup F)} = \frac{a+b}{a+b+c}$$

and the right hand side is

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{b}{b+c}$$

Cross multiplying (since all numbers are non-negative) we see that the inequality is equivalent to

$$ab + b^2 + ac + bc = (a+b)(b+c) \geq b(a+b+c) = ab + b^2 + 0 + bc$$

Since $ac \geq 0$ the inequality is valid.

---

c) This is also true. Since E & F are independent so are E^c & F as are E & F^c hence we have

$$\begin{aligned} P(E^c \cup F) - P(F) &= (1 - P(EF^c)) - P(F) \\ &= 1 - P(E)(1 - P(F)) - P(F) \\ &= 1 - P(E) + P(E)P(F) - P(F) \\ &= 1 - P(F)(1 - P(E)) - P(E) \\ &= (1 - P(E^cF)) - P(E) \\ &= P(E \cup F^c) - P(E) \end{aligned}$$

(Actually, independence is not needed & drawing a Venn diagram both sides can be seen to equal $P((E \cup F)^c)$.)

Problem 4 (20 points). Suppose X is a discrete random variable whose distribution function is given by

$$F(t) = P(X \le t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/4 & \text{if } 0 \le t < 1 \\ 1/2 & \text{if } 1 \le t < 2 \\ 2/3 & \text{if } 2 \le t < 3 \\ 1 & \text{if } 3 \le t \end{cases}$$

a) Compute the expectation and variance of X.

b) Compute the distribution function for the random variable $Y = X^2$.

a) Let's first compute the mass function of X:

$$p(0) = \frac{1}{4}, \quad p(1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \quad p(2) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}, \quad p(3) = 1 - \frac{2}{3} = \frac{1}{3}$$

We can now compute the expectation

$$E[X] = 0\cdot\left(\frac{1}{4}\right) + 1\cdot\left(\frac{1}{4}\right) + 2\cdot\left(\frac{1}{6}\right) + 3\cdot\left(\frac{1}{3}\right) = \boxed{\frac{19}{12}}$$

Also $E[X^2] = 0^2\cdot\left(\frac{1}{4}\right) + 1^2\cdot\left(\frac{1}{4}\right) + 2^2\cdot\left(\frac{1}{6}\right) + 3^2\cdot\left(\frac{1}{3}\right) = \frac{47}{12}$

Hence $Var(X) = E[X^2] - (E[X])^2 = \frac{47}{12} - \left(\frac{19}{12}\right)^2 = \boxed{\frac{203}{144}}$

---

b) Let's now compute the mass function for X^2:

$$p(0^2) = \frac{1}{4}, \quad p(1^2) = \frac{1}{4}, \quad p(2^2) = \frac{1}{6}, \quad p(3^2) = \frac{1}{3}$$

Hence the distribution function is:

$$\boxed{F_{X^2}(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 1/4 & \text{if } 0 \le t < 1 \\ 1/2 & \text{if } 1 \le t < 4 \\ 2/3 & \text{if } 4 \le t < 9 \\ 1 & \text{if } 9 \le t \end{cases}}$$

Problem 5 (20 points). In the city of Nashville, approximately 30 automobiles are stolen every week. Modeling this situation with a Poisson random variable, find the probability that no more than 1 automobile will be stolen in Nashville today.

Since the rate of stolen cars is $30/\text{week} = 30/7 / \text{day}$

we have a Poisson random variable X with parameter $\lambda = \frac{30}{7}$

$$\therefore P(X \leq 1) = p(0) + p(1) = e^{-\lambda} + e^{-\lambda}\frac{\lambda^1}{1!}$$

$$= e^{-\frac{30}{7}} + \frac{30}{7}e^{-\frac{30}{7}}$$

$$= \frac{37}{7}e^{-\frac{30}{7}}$$