HOMEWORK 2, MATH 362A - FALL 2010

DUE MONDAY SEPTEMBER 20TH

For this homework assignment we will use ideas involving the Hahn-Banach Theorem to introduce amenable groups. Recall that a countable discrete group G is amenable if there exists a finitely additive probability measure μ on 2^G such that $\mu(g \cdot E) = \mu(E)$ for all $E \subset G$.

1. Suppose $\mathcal{X}$ is a vector space over $\mathbb{F}$ and $P \subset \mathcal{X}$ is a set such that for all $x, y \in P$, and $\alpha > 0$ we have $x + y \in P$ and $\alpha x \in P$. We write $x \leq y$ if $y - x \in P$. A linear functional f on $\mathcal{X}$ is said to be positive if $f(x) \geq 0$ for all $x \in P$.

Show that if $S \subset \mathcal{X}$ is a linear subspace such that for all $x \in \mathcal{X}$ there exists $s \in S$ such that $x \leq s$. Then any positive linear functional on S extends to a positive linear functional on $\mathcal{X}$.

2. Let $\mathcal{I}$ be a non-empty countable set. Show that given a finitely additive probability measure μ on $2^{\mathcal{I}}$, there exists a unique continuous linear functional $\phi_\mu \in \ell^\infty(\mathcal{I})^*$ such that $\phi_\mu(\chi_E) = \mu(E)$ for every subset $E \subset \mathcal{I}$. Conversely, show that if $\phi \in \ell^\infty(\mathcal{I})^*$ such that $\phi(f) \geq 0$, for all $f \in \ell^\infty(\mathcal{I})$, $f \geq 0$, and $\phi(1) = 1$, then $\mu(E) = \phi(\chi_E)$ defines a finitely additive probability measure on $\mathcal{I}$.

3. Let $0 \to H \to G \to K \to 0$ be a short exact sequence of groups. Show that G is amenable if and only if both H and K are amenable.

4. Suppose G_i is an increasing sequence of amenable groups, show that $G = \cup_{i \in \mathbb{N}} G_i$ is also amenable.

5. Show that all finite, and abelian groups are amenable.