HOMEWORK 1, MATH 362A - FALL 2010

DUE WEDNESDAY SEPTEMBER 8TH

1. Let $\mathcal{H}$ be a Hilbert space. Show that if $X \subset \mathcal{H}$ is a bounded set then X has a unique Chebyshev center, i.e., there exists a unique element $h_0 \in \mathcal{H}$ such the function $h \mapsto \sup_{x \in X} \|h - x\|$ obtains its minimum at h_0.

2. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, show that any Hamel (algebraic) basis for $\mathcal{H}$ must be uncountable.

3. Let B be a Banach space with norm $\|\cdot\|$. Show that B is isometrically isomorphic to a Hilbert space if and only if the parallelogram identity holds: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in B$.

4. Show that an inner-product space H is complete if and only if any maximally orthogonal set is an orthogonal basis.

(**Bonus Problem**). From the Gram-Schmidt process we know that if $\mathcal{H}$ is a separable Hilbert space and $V \subset \mathcal{H}$ is a dense subspace then V contains an orthonormal basis for $\mathcal{H}$. Show that this is not true in general.