HOMEWORK 9, MATH 175 - FALL 2009

This homework assignment covers Sections 17.1-17.4 in the book.

1. Sketch the vector field $F(x,y) = \frac{1}{x}\mathbf{i} + y\mathbf{j}$.

2. Find the gradient vector field for $f(x,y) = x^2 - y$ and sketch it.

The gradient vector field is just $\boxed{\nabla f(x,y) = 2x\mathbf{i} - \mathbf{j}.}$

3. Evaluate the line integral $\int_C x \sin y \ ds$ where C is the line segment from $(0,3)$ to $(4,6)$.

The curve C can be parametrized by $r(t) = (0,3) + t(4,3)$ where $0 \le t \le 1$, and then we have $\|r'(t)\| = \sqrt{4^2+3^2} = 5$. Hence

$$\int_C x \sin y \ ds = \int_0^1 20t \sin(3+3t) \ dt,$$

integration by parts ($u = t$ and $dv = \sin(3+3t)dt$) then gives

$$\int_0^1 20t \sin(3+3t) \ dt = 20[-\frac{1}{3}t\cos(3+3t) + \frac{1}{9}\sin(3+3t)]_0^1$$

$$= \boxed{\frac{20}{9}(\sin 6 - 3\cos 6 - \sin 3).}$$

4. Evaluate the line integral $\int_C \sin x \ dx + \cos y \ dy$, where C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$ and the line segment from $(-1,0)$ to $(-2,3)$.

If we split the curve into two parts we can find a parameterization for each part and then continue as in 3. Let's instead use the Fundamental Theorem of Line Integrals.

Note that $F(x,y) = \sin x\mathbf{i} + \cos y\mathbf{j}$ is a conservative vector field. Indeed if $f_x = \sin x$ then $f = -\cos x + g(y)$ where g is a function of y. Then we have $\cos y = f_y = g'(y)$ so that $g(y) = \sin y + K$ where K is some constant.

In particular we have $F = \nabla(-\cos x + \sin y)$ and hence we have

$$\int_C \sin x \ dx + \cos y \ dy = \int_C \nabla f \cdot dr$$

$$= f(-2,3) - f(1,0) = \boxed{-\cos 2 + \sin 3 + \cos 1.}$$

5. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $F(x,y,z) = (x+y)\mathbf{i} + (y-z)\mathbf{j} + z^2\mathbf{k}$ and C is given by the vector function $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$.

F is not a conservative vector field and so we cannot use the Fundamental Theorem of Line Integrals. We will have to compute this directly. Since $r(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$ we have $r'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + 2t\mathbf{k}$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F(r(t)) \cdot r'(t) dt$$

$$= \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5)dt = \int_0^1 (5t^5 - t^4 + 2t^3)dt$$

$$= \frac{5}{6} - \frac{1}{5} + \frac{1}{2} = \boxed{\frac{17}{15}.}$$

6. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $F(x,y,z) = (2xz + y^2)\mathbf{i} + 2xy\mathbf{j} + (x^2 + 3z^2)\mathbf{k}$ and C is given by $x = t^2, y = t+1, z = 2t-1,\ 0 \le t \le 1$.

If this is a conservative vector field then we have

$$f_x = 2xz + y^2,$$

hence $f = x^2 z + xy^2 + g(y,z)$ where g is some function. Therefore we have

$$2xy = f_y = 2xy + \partial g/\partial y,$$

and so $g(y,z) = h(z)$ for some function h. We then have $f = x^2 z + xy^2 + h(z)$ and so

$$x^2 + 3z^2 = f_z = x^2 + h'(z),$$

hence $h(z) = z^3 + K$ for some constant K.

In particular we have shown that $F = \nabla(x^2 z + xy^2 + z^3)$ and so by the Fundamental Theorem of Line Integrals we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,1) - f(0,1,-1) = (1+4+1) - (0+0-1) = \boxed{7.}$$

7. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $F(x,y,z) = e^y\mathbf{i} + xe^y\mathbf{j} + (z+1)e^z\mathbf{k}$, and C is given by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \le t \le 1$.

Just as above, if F is a conservative vector field then we have

$$f_x = e^y,$$

hence $f = xe^y + g(y,z)$. Therefore

$$xe^y = f_y = xe^y + \partial g/\partial y,$$

and so $g(y,z) = h(z)$. We have then $f = xe^y + h(z)$ and so

$$(z+1)e^z = f_z = h'(z),$$

therefore $h(z) = ze^z + K$ and in particular we have $F = \nabla(xe^y + ze^z)$ and so by the Fundamental Theorem of Line Integrals we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = \boxed{2e.}$$

8. Evaluate the line integral $\int_C \cos y\ dx + x^2 \sin y\ dy$, where C is the rectangle with vertices $(0,0)$, $(5,0)$, $(5,2)$, and $(0,2)$ oriented positively.

Let D be the region enclosed by the curve C. Using Green's Theorem we have that

$$\int_C \cos y\ dx + x^2 \sin y\ dy = \iint_D (2x \sin y + \sin y)\ dA$$

$$= \int_0^5 \int_0^2 (2x+1) \sin y\ dy\ dx = [x^2 + x]_0^5 [-\cos y]_0^2 = \boxed{30(1 - \cos 2).}$$

9. Evaluate the line integral $\int_C \sin y\ dx + x \cos y\ dy$, where C is given by the ellipse $x^2 + xy + y^2 = 1$, oriented positively.

Let D be the region enclosed by the curve C. Using Green's Theorem we have that

$$\int_C \sin y\ dx + x \cos y\ dy = \iint_D (\cos y - \cos y) dA = \boxed{0.}$$