Math 196 - Exam 1, September 23, 2008 Name:________Key________

Problem 1 (15 points). Give an explicit solution (or show that one does not exist) for the differential equation given by $y^2\frac{d^2y}{dx^2} = \frac{dy}{dx}$, under the initial conditions $y(0) = 2$, $\frac{dy}{dx}(0) = \frac{-1}{2}$.

Solution 1. Substituting $v = \frac{dy}{dx}$ wee see that $\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$. We therefore obtain a new equation:

$$y^2 v\frac{dv}{dy} = v.$$

This is easily solved by separation to obtain $v = -\frac{1}{y} + C_1$, where C_1 is a constant. Setting $x = 0$ in this formula and using the initial value conditions above we see that

$$\frac{-1}{2} = \frac{dy}{dx}(0) = -\frac{1}{y(0)} + C_1 = -\frac{1}{2} + C_1,$$

so that $C_1 = 0$.

Therefore $\frac{dy}{dx} = v = -\frac{1}{y}$ and so again by separation we obtain

$$\frac{1}{2}y^2 = -x + C_2,$$

for some constant C_2. From the initial conditions we obtain $\frac{1}{2}2^2 = 0 + C_2$, so $C_2 = 2$ and solving for y we have

$$y = \sqrt{4 - 2x}.$$

Problem 2 (20 points). Suppose we have a 100 gallon tub which is full with a salt water solution which contains 10 pounds of salt. Suppose we add pure water to the tub at the rate of 1 gallon per minute while we take away from the solution at the same rate.

(1). Assuming that the tub is well stirred, how much salt remains in the pot after 1 hour?

(2). Suppose we repeat the above experiment, but in addition we boil the solution so that (pure) water is evaporated from the tub at the rate of $\frac{1}{10}$ gallons per minute. Then how much salt remains after 1 hour?

Solution 2. We know that the change in the amount of salt ΔX over a period of time Δt is just the amount we gain minus the amount we lose. This leads to the differential equation

$$\frac{dX}{dt} = r_i c_i - \frac{r_o X}{V_0 + (r_i - r_o)t},$$

where r_i and c_i (r_o and c_o) are the rate and concentration of salt for the solution coming in (going out), and V_0 is the initial volume of solution in the tub.

(1). Substituting in the values ($r_i = r_o = 1$, $c_i = 0$, $V_0 = 100$) in the above equation we have

$$\frac{dX}{dt} = -\frac{X}{100}.$$

Solving by separation gives the formula for X:

$$X(t) = Ce^{-t/100}.$$

We know $X(0) = 10$ and so we see that $C = 10$, and hence the amount of salt (in pounds) left after an hour is

$$X(60) = 10e^{-60/100} = 10e^{-3/5}.$$

(2). Here we have an additional amount ($\frac{1}{10}$ pounds per minute) of pure water which is leaving the tub. Perhaps the easiest way to view this is to notice that we then have a net input of $\frac{9}{10}$ pounds per minute of pure water. Thus we can view this as the same situation in part (1) with the only difference being that $r_i = \frac{9}{10}$. The new differential equation is then:

$$\frac{dX}{dt} = -\frac{X}{100 - ((9/10) - 1)t} = -\frac{10X}{1000 - t}.$$

Again this is separable and so we find our solution for X is given by:

$$X(t) = C(1000 - t)^{10},$$

where C is a constant. From the initial condition $X(0) = 10$ we see that $C = 10/1000^{10}$, hence the amount of salt (in pounds) left after an hour is

$$X(60) = (10/1000^{10})(1000 - 60)^{10} = 10(940/1000)^{10} = 10(0.94)^{10}.$$

Problem 3 (20 points). Give an implicit solution (or show that one does not exist) for the differential equation given by $y(1-x^2)\frac{dy}{dx} = e^{2x}(1+2x-x^2) + xy^2$, under the initial condition $y(0) = 2$.

Solution 3. If we rewrite the above equation in the form:

$$y(1-x^2)dy - (e^{2x}(1+2x-x^2) + xy^2)dx = 0$$

then we see that

$$\frac{\partial}{\partial x}(y(1-x^2)) = -2xy = \frac{\partial}{\partial x}(-(e^{2x}(1+2x-x^2) + xy^2))$$

and thus the equation is exact.

Therefore an implicit solution exists of the form $F(x,y) = C$ for some function $F : \mathbb{R}^2 \to \mathbb{R}$. We know from the above equation that we have $\frac{\partial F}{\partial y} = y(1-x^2)$ and hence

$$F(x,y) = \int y(1-x^2)dy = \frac{1}{2}y^2(1-x^2) + h(x),$$

for some function $h : \mathbb{R} \to \mathbb{R}$.

Differentiating with respect to x gives $-(e^{2x}(1+2x-x^2) + xy^2) = -xy^2 + h'(x)$ so that

$$h'(x) = -e^{2x}(1+2x-x^2),$$

Integrating using integration by parts (sorry folks, my original problem had a bunch of cancellation but I made some typos and so it turned into a mess) gives $h(x) = \frac{-1}{4}(2x^2e^{2x} + 2xe^{2x} - 3e^{2x})$, and so

$$C = F(x,y) = \frac{1}{2}y^2(1-x^2) + \frac{-1}{4}(2x^2e^{2x} + 2xe^{2x} - 3e^{2x}).$$

From the initial condition we see that $C = F(0,2) = \frac{11}{4}$ and so the implicit solution is given by

$$\frac{11}{4} = \frac{1}{2}y^2(1-x^2) + \frac{-1}{4}(2x^2e^{2x} + 2xe^{2x} - 3e^{2x}).$$

Problem 4 (10 points). The general solution to the differential equation $y''' - 2y'' - y' + 2y = 0$ is given by the function $y = Ae^x + Be^{-x} + Ce^{2x}$, where A, B, and C are constants. Given the initial conditions $y(0) = 4, y'(0) = -1$ and $y''(0) = 1$, determine what the constants A, B, and C are.

Solution 4. Since $y = Ae^x + Be^{-x} + Ce^{2x}$ we have that $y' = Ae^x - Be^{-x} + 2Ce^{2x}$ and $y'' = Ae^x + Be^{-x} + 4Ce^{2x}$. Hence the initial conditions give us the system of linear equations:

$$A + B + C = y(0) = 4$$

$$A - B + 2C = y'(0) = -1$$

$$A + B + 4C = y''(0) = 1$$

Subtracting the first equation from the second and third gives the new system of linear equations:

$$A + B + C = 4$$

$$-2B + C = -5$$

$$3C = -3$$

Dividing the third equation by 3 and then subtracting this from the first two equations gives:

$$A + B = 5$$

$$-2B = -4$$

$$C = -1$$

Dividing the second equation by -2 and then subtracting this from the first equation gives the solution:

$$A = 3$$

$$B = 2$$

$$C = -1$$

Problem 5 (15 points). Give an explicit general solution (or show that one does not exist) for the differential equation given by $\frac{dy}{dx} = \frac{-y}{x} + \frac{1}{e^{xy}}$.

Solution 5. Substituting $v = xy$ gives us that $y = v/x$ and $\frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2} = \frac{dy}{dx}$, hence we have the new differential equation

$$\frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2} = \frac{dy}{dx} = \frac{-v}{x^2} + \frac{1}{e^v},$$

i.e. $\frac{dv}{dx} = \frac{x}{e^v}$.

Solving this by separation we have $e^v = \frac{1}{2}x^2 + C$, for some constant C. Setting $v = xy$ and then solving for y gives the solution

$$y = \frac{1}{x}\ln(\frac{1}{2}x^2 + C).$$

Problem 6 (20 points). Suppose an object is on a flat frictionless slide which is tilted to have slope b, (note in the diagram $b < 0$). Let $X(t)$ and $Y(t)$ be respectively the x-coordinate and y-coordinate of the object at time t.

By Newton's third law of motion we know that the force caused by gravity results in the following diagram:

(1). From this information, derive a second order differential equation relating $X(t)$ to the slope of the slide. (Hint: Using the diagrams above, calculate $\tan\theta$ in multiple ways.)

(2). Suppose that the slide is given by the equation $y = -x + 1$ (in meters), and that the object is placed at rest at the point $(0, 1)$. How long does it take for the object to reach the point $(1, 0)$?

(3). Suppose instead of a flat slide, we have a curve given by the equation $y = 4 - x^2$, give an explicit differential equation for $X(t)$ which describes this situation. (You do not have to solve this differential equation.)

Solution 6. (1). Using the fact that for an angle θ in a triangle, $\tan\theta$ is opposite over adjacent we can compute $\tan\theta$ in three different ways:

$$\tan\theta = -\frac{1}{b}, \quad \tan\theta = \frac{X''}{-Y''}, \quad \text{and} \quad \tan\theta = \frac{9.8 + Y''}{X''}.$$

(Note b and Y'' are negative values while X'' is a positive value and hence $\tan\theta$ is positive.)

From the above equations, solving for Y'' on one hand gives $Y'' = bX''$, while solving for X'' on the other hand gives $X'' = -b(9.8 + Y'')$. Substituting $Y'' = bX''$ in the second equation gives $X'' = -b(9.8 + bX'')$. Solving for X'' gives the second order differential equation

$$X''(t) = -\frac{9.8b}{1 + b^2}.$$

(2). For the equation $y = -x + 1$ the slope of the slide is $b = -1$ and hence the above equation becomes:

$$X''(t) = \frac{9.8}{2} = 4.9.$$

Solving for the horizontal velocity X' gives $X'(t) = 4.9t + C_1$, however the object begins at rest and so $C_1 = 0$. Thus the horizontal position X of the object is $X(t) = 2.45t^2 + C_2$, however the object starts at the point $X(0) = 0$ so that $C_2 = 0$, and so

$$X(t) = 2.45t^2.$$

We want to know when the object reaches the point $(1, 0)$, i.e. when is $X(t) = 1$. Setting $X(t) = 1$ and solving for t gives

$$t = \frac{1}{\sqrt{2.45}} \text{ seconds.}$$

(3). For the curve $y = 4 - x^2$ we may regard the object at each point on the curve as being on an "infinitesimal slide" with the slope of the slide being the slope of the tangent line, i.e. at the point x the slope of the "infinitesimal slide" is given by $\frac{dy}{dx} = -2x$.

Therefore the differential equation for $X(t)$ which describes this situation is given by:

$$X''(t) = \frac{-9.8\frac{dy}{dx}}{1 + (\frac{dy}{dx})^2} = \frac{-19.6X}{1 + 4X^2}.$$