Math 196 - Exam 2, October 14, 2008 Name:____

Problem 1 (15 points). Find all solutions to the following system of linear equations, check your work:

$$x_1 + x_2 - x_3 = 0$$

$$2x_2 - 2x_3 = -2$$

$$x_1 - x_2 + x_3 = 2.$$

Problem 2 (10 points).

1. Show that the set of functions $y(x)$ which satisfy the differential equation $y'' - 2y' + y = 0$ form a subspace of the vector space of all functions.

2. Using part 1 and the fact that the functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions to the above differential equation, find a solution $y(x)$ such that $y(0) = 1$ and $y(1) = 0$.

Problem 3 (20 points). Find a basis and calculate the dimension for each of the following vector spaces V:

1. $V = \mathbb{R}^5$.
2. $V = \{0\}$.
3. V is the set of all 2×2 matrices A such that $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
4. V is the set of polynomials p of degree at most 3 such that $p(0) = p(1) = 0$.
5. V is the solution space to the following homogeneous system of linear equations:

$$x + z = 0$$

$$y - z = 0$$

$$x - y + 2z = 0.$$

Problem 4 (20 points). Determine whether or not the following set forms a basis for $\mathbb{R}^3$, justify your answer.

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} \right\}.$$

Problem 5 (20 points). Determine wether or not the following are vector spaces, if they are then calculate the dimension, if they are not then give a reason why not:

1. V is the set of points $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ such that $x = y$ with the vector space structure coming from $\mathbb{R}^2$.

2. V is the set of points $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that $xy = yz$ with the vector space structure coming from $\mathbb{R}^3$.

3. V is the set of convergent sequences $\{x_n\}_n$ with the zero vector given by the zero sequence $\vec{0} = \{0\}_n$, addition of vectors given by entrywise addition $\{x_n\}_n \oplus \{y_n\}_n = \{x_n + y_n\}_n$, and scalar multiplicatoin given by entrywise multiplication $a \cdot \{x_n\}_n = \{ax_n\}_n$.

4. V is the set of positive real numbers $r > 0$ with the zero vector given by $\vec{0} = 1$, addition of vectors given by $r_1 \oplus r_2 = r_1 r_2$, and scalar multiplication given by $a \cdot r = r^a$.

5. V is the set of all linear operators $T : \mathbb{R}^2 \to \mathbb{R}$ with the zero vector given by the constant zero function $\vec{0} = 0$, addition of vectors given by pointwise addition $(T \oplus S)(v) = Tv + Sv$, and scalar multiplication given by pointwise multiplication $(a \cdot T)(v) = a(Tv)$.

Problem 6 (15 points). Recall from calculus the power series expansions for e^x, $\sin x$, and $\cos x$:

$$e^x = \Sigma_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

$$\sin x = \Sigma_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = \Sigma_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Consider the matrix $A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$.

1. Calculate A, A^2, A^3 and A^4.
2. Given a natural number $n \in \mathbb{N}$ calculate A^{2n} and A^{2n+1}.
3. Calculate $e^A = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$.