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On cocycle superrigidity for Gaussian actions

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Abstract. We present a general setting to investigate $\mathcal{U}_{\text{fin}}$-cocycle superrigidity for Gaussian actions in terms of closable derivations on von Neumann algebras. In this setting we give new proofs to some $\mathcal{U}_{\text{fin}}$-cocycle superrigidity results of S. Popa and we produce new examples of this phenomenon. We also use a result of K. R. Parthasarathy and K. Schmidt to give a necessary cohomological condition on a group representation in order for the resulting Gaussian action to be $\mathcal{U}_{\text{fin}}$-cocycle superrigid.

1. *Introduction*

A central motivating problem in the theory of measure-preserving actions of countable groups on probability spaces is to classify certain actions up to orbit equivalence, i.e. isomorphism of the underlying probability spaces such that the orbits of one group are carried onto the orbits of another. When the groups are amenable, this problem was completely settled in the early 1980s (cf. [**9, 13, 14, 23**]): all free ergodic actions of countable, discrete, amenable groups are orbit equivalent. The non-amenable case, however, is much more complex and has recently seen a flourish of activity including a number of striking results. We direct the reader to the survey articles [**39, 49**] for a summary of these recent developments.

One breakthrough which we highlight here is Popa's use of his deformation/rigidity techniques in von Neumann algebras to produce rigidity results for orbit equivalence (cf. [**33, 35–38, 40, 41**]). One of the seminal results using these techniques is Popa's cocycle superrigidity theorem [**38, 40**] (see also [**15, 51**] for more on this), which obtains orbit equivalence superrigidity results by means of untwisting cocycles into a finite von Neumann algebra. In order to state this result, we recall a few notions regarding groups.

A subgroup $\Gamma_0 \subset \Gamma$ is *wq-normal* if there exists no intermediate subgroup $\Gamma_0 \subset K \subsetneq \Gamma$ such that $\gamma K \gamma^{-1} \cap K$ is finite for all $\gamma \in \Gamma \backslash K$. If $\mathcal{U}$ is a class of Polish groups, then a free, ergodic, measure-preserving action of a countable discrete group Γ on a standard probability space (X, μ) is said to be $\mathcal{U}$*-cocycle superrigid* if any cocycle for the action $\Gamma \curvearrowright (X, \mu)$ which is valued in a group contained in the class $\mathcal{U}$ must be cohomologous to a homomorphism. $\mathcal{U}_{\text{fin}}$ is used to denote the class of Polish groups which arise as closed subgroups of the unitary groups of II_1 factors. In particular, the class of compact Polish groups and the class of countable discrete groups are both contained in $\mathcal{U}_{\text{fin}}$. The notions of wq-normality and the class $\mathcal{U}_{\text{fin}}$ are due to Popa (cf. [**35, 38**]).

POPA'S COCYCLE SUPERRIGIDITY THEOREM. ([**38, 40**] (for Bernoulli shift actions)) *Let Γ be a group which contains an infinite wq-normal subgroup Γ_0 such that the pair (Γ, Γ_0) has relative property (T), or such that Γ_0 is the direct product of an infinite group and a non-amenable group, and let (X_0, μ_0) be a standard probability space. Then the Bernoulli shift action $\Gamma \curvearrowright \Pi_{g\in\Gamma}(X_0, \mu_0)$ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

The proof of this theorem uses a combination of deformation/rigidity and intertwining techniques that were initiated in [**34**]. Roughly, if we are given a cocycle into a unitary group of a II_1 factor, we may consider the 'twisted' group algebra sitting inside the group-measure space construction. The existence of rigidity can then be contrasted against natural malleable deformations from the Bernoulli shift in order to locate the 'twisted' algebra inside the group-measure space construction. Locating the 'twisted' algebra allows us to 'untwist' it and, in so doing, untwist the cocycle in the process.

The existence of such s-malleable deformations (introduced by Popa in [**36, 37**]) actually occurs in a broader setting than the (generalized) Bernoulli shifts with diffuse core, but it was Furman [**15**] who first noticed that the even larger class of Gaussian actions are also s-malleable. The class of Gaussian actions has a rich structure, owing to the fact the every Gaussian action of a group Γ arises functorially from an orthogonal representation of Γ. The interplay between the representation theory and the ergodic theory of a group via the Gaussian action has been fruitfully exploited in the literature (cf. the seminal works of Connes and Weiss and of Schmidt [**10, 47, 48**] *inter alios*).

In this paper, we will explore $\mathcal{U}_{\text{fin}}$-cocycle superrigidity within the class of Gaussian actions. An advantage to our approach is that we develop a general framework for investigating cocycle superrigidity of such actions by using derivations on von Neumann algebras. The first theme we take up is the relation between the cohomology of group representations and the cohomology of their respective Gaussian actions. Under general assumptions, we show that cohomological information coming from the representation can be faithfully transferred to the cohomology group of the action with coefficients in the circle group $\mathbb{T}$. As a consequence, we obtain our first result, that the cohomology of the representation provides an obstruction to the $\mathcal{U}_{\text{fin}}$-cocycle superrigidity of the associated Gaussian action.

THEOREM 1.1. *Let Γ be a countable discrete group and $\pi : \Gamma \to \mathcal{O}(\mathcal{K})$ a weakly mixing orthogonal representation. A necessary condition for the corresponding Gaussian action to be $\{\mathbb{T}\}$-cocycle superrigid is that $H^1(\Gamma, \pi) = \{0\}$.*

The Bernoulli shift action of a group is precisely the Gaussian action corresponding to the left regular representation, and the circle group $\mathbb{T}$ is contained in the class $\mathcal{U}_{\text{fin}}$. When combined with [**31**, Corollary 2.4], which states that for a non-amenable group vanishing of the first ℓ^2-Betti number is equivalent to $H^1(\Gamma, \lambda) = \{0\}$, we obtain the following corollary.

COROLLARY 1.2. *Let Γ be a countable discrete group. If $\beta_1^{(2)}(\Gamma) \neq 0$, then the Bernoulli shift action is not $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

The second theme explored is the deformation/derivation duality developed by the first author in [**29**]. The flexibility inherent at the infinitesimal level allows us to offer a unified treatment of Popa's theorem in the case of generalized Bernoulli actions and expand the class of groups whose Bernoulli actions are known to be $\mathcal{U}_{\text{fin}}$-cocycle superrigid. As a partial converse to the above results, we have that an *a priori* stronger property than having $\beta_1^{(2)}(\Gamma) = 0$, L^2-rigidity (see Definition 2.13) is sufficient to guarantee $\mathcal{U}_{\text{fin}}$-cocycle superrigidity of the Bernoulli shift. For this result, and throughout this paper, we denote by $L\Gamma$ the group von Neumann algebra of Γ, i.e. $L\Gamma$ is the smallest von Neumann algebra in $\mathcal{B}(\ell^2\Gamma)$ which contains the image of the left regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$.

THEOREM 1.3. *Let Γ be a countable discrete group. If $L\Gamma$ is L^2-rigid, then the Bernoulli shift action of Γ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

Examples of groups for which this holds are groups which contain an infinite normal subgroup which has relative property (T) or is the direct product of an infinite group and a non-amenable group, recovering Popa's cocycle superrigidity theorem for Bernoulli actions of these groups.

We also obtain new groups for which Popa's theorem holds. For example, we show that the theorem holds for any generalized wreath product $A_0 \wr_X \Gamma_0$, where A_0 is a non-trivial abelian group and Γ_0 does not have the Haagerup property. Also, if $L\Lambda$ is non-amenable and has property Gamma of Murray and von Neumann [**22**], then the theorem also holds for Λ.

We remark that it is still an open question whether vanishing of the first ℓ^2-Betti number characterizes groups whose Bernoulli actions are $\mathcal{U}_{\text{fin}}$-cocycle superrigid. For instance, it is still unknown for the group $\mathbb{Z} \wr \mathbb{F}_2$, which contains an infinite normal abelian subgroup and hence has vanishing first ℓ^2-Betti number by [**5**].

2. *Preliminaries*

We begin by reviewing the basic notions of Gaussian actions, cohomology of representations and actions, and closable derivations. Though our treatment of the last two topics is standard, our approach to Gaussian actions is somewhat non-standard, where we take a more operator-algebraic approach by viewing the algebra of bounded functions on the probability space as a von Neumann algebra acting on a symmetric Fock space. In the non-commutative setting of free probability, this is the same as Voiculescu's approach in [**52**]. But, first, let us recall a few basic definitions and concepts which constitute the basic language in which this paper is written. Throughout, all Hilbert spaces are assumed to be separable.

Definition 2.1. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation and denote by π^{op} the associated contragredient representation on the contragredient Hilbert space $\mathcal{H}^{\text{op}}$ of $\mathcal{H}$. We say that π:

(1) is *ergodic* if π has no non-zero invariant vectors;

(2) is *weakly mixing* if $\pi \otimes \pi^{\text{op}}$ is ergodic (equivalently, $\pi \otimes \rho^{\text{op}}$ is ergodic for any unitary Γ-representation ρ);

(3) is *mixing* if $\langle \pi_\gamma(\xi), \eta \rangle \to 0$ as $\gamma \to \infty$ for all $\xi, \eta \in \mathcal{H}$;

(4) has a *spectral gap* if there exists $K \subset \Gamma$, finite, and $C > 0$ such that

$$\|\xi - P(\xi)\| \leq C \sum_{k \in K} \|\pi_k(\xi) - \xi\| \quad \text{for all } \xi \in \mathcal{H},$$

where P is the projection onto the invariant vectors;

(5) has a *stable spectral gap* if $\pi \otimes \pi^{\text{op}}$ has a spectral gap (equivalently, $\pi \otimes \rho^{\text{op}}$ has a spectral gap for any unitary Γ-representation ρ);

(6) is *amenable* if π is either not weakly mixing or does not have a stable spectral gap.

Note that for an orthogonal representation π of Γ into a real Hilbert space $\mathcal{K}$, the associated unitary representation into $\mathcal{K} \otimes \mathbb{C}$ is canonically isomorphic to its contragredient. Hence, in this situation we may replace in the above definition '$\pi \otimes \pi^{\text{op}}$' and '$\pi \otimes \rho^{\text{op}}$' with '$\pi \otimes \pi$' and '$\pi \otimes \rho$', respectively.

Let $\Gamma \curvearrowright^{\sigma} (X, \mu)$ be an action of the countable discrete group Γ by μ-preserving automorphisms of a standard probability space (X, μ). This yields a unitary representation $\pi^{\sigma} : \Gamma \to \mathcal{U}(L^2_0(X, \mu))$ called the *Koopman representation* associated to σ. (Here $L^2_0(X, \mu)$ denotes the orthogonal complement in $L^2(X, \mu)$ to the subspace of the constant functions on X.) Note that the Koopman representation is the unitary representation associated to the orthogonal representation of Γ acting on the real-valued L^2-functions. We say that the action σ is ergodic (or weakly mixing, mixing, etc.) if the Koopman representation π^{σ} is in the sense of the above definition. An action $\Gamma \curvearrowright^{\sigma} (X, \mu)$ is *(essentially) free* if, for all $\gamma \in \Gamma$, $\gamma \neq e$, $\mu\{x \in X : \sigma_{\gamma}(x) = x\} = 0$.

Given unitary representations $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$, we say that π is *contained* in ρ if there is a linear isometry $V : \mathcal{H} \to \mathcal{K}$ such that $\pi_{\gamma} = V^* \rho_{\gamma} V$ for all $\gamma \in \Gamma$. We say that π is *weakly contained* in ρ if for any $\xi \in \mathcal{H}$, $F \subset \Gamma$ finite, and $\varepsilon > 0$, there are $\xi'_1, \ldots, \xi'_n \in \mathcal{K}$ such that

$$\left| \langle \pi_{\gamma}(\xi), \xi \rangle - \sum_{k=1}^{n} \langle \rho_{\gamma}(\xi'_k), \xi'_k \rangle \right| < \varepsilon \quad \text{for all } \gamma \in F.$$

Note that amenability of a representation π is equivalent to $\pi \otimes \pi^{\text{op}}$ weakly containing the trivial representation, which is equivalent with Bekka's definition by [**1**, Theorem 5.1].

A finite von Neumann algebra is a von Neumann algebra, possessing a normal faithful state τ, which is also a trace, i.e. $\tau(xy) = \tau(yx)$ for all $x, y \in M$. Throughout this paper, we will assume that a finite von Neumann algebra comes with a fixed trace, and by an inclusion of finite von Neumann algebras $(M, \tau) \subset (\tilde{M}, \tilde{\tau})$ we mean an inclusion $M \subset \tilde{M}$ such that $\tilde{\tau}$ is a trace on $\tilde{M}$ which agrees with τ when restricted to M. A finite von Neumann algebra is a factor if the center consists of scalar multiples of the identity, or equivalently if the trace is unique. A II_1 factor is a finite factor which is not finite dimensional.

For a finite von Neumann algebra (M, τ), the trace τ induces a positive-definite sesquilinear form on M given by $\langle x, y \rangle = \tau(y^* x)$. We denote by $L^2(M, \tau)$ the Hilbert space completion of M with respect to this form. The multiplication structure on M induces a normal M–M bimodule structure on $L^2(M, \tau)$. We denote by $\mathcal{U}(M)$ the group of unitaries of M, and by $(M)_1$ the unit ball of M with respect to the operator norm.

Associated to a measure-preserving action $\Gamma \curvearrowright^{\sigma} (X, \mu)$ of a countable discrete group Γ on a probability space (X, μ) is a finite von Neumann algebra known as the group-measure space construction [**21**]. Note that Γ acts on $L^{\infty}(X, \mu)$ (we will also denote this action by σ) by the formula $\sigma_\gamma(f) = f \circ \sigma_{\gamma^{-1}}$ and, since the action of Γ on X preserves the measure, this action on $L^\infty(X, \mu)$ preserves the integral.

Consider the Hilbert space

$$\mathcal{H} = \ell^2(\Gamma, L^2(X, \mu)) = \{\Sigma_{\gamma\in\Gamma} a_\gamma u_\gamma \mid a_\gamma \in L^2(X, \mu),\ \Sigma_{\gamma\in\Gamma} \|a_\gamma\|_2^2 < \infty\},$$

where u_γ denotes the function on Γ which is 1 at γ and 0 elsewhere.

On this Hilbert space, we define a convolution operation by

$$(\Sigma_{\gamma\in\Gamma} a_\gamma u_\gamma) \cdot (\Sigma_{\lambda\in\Gamma} b_\lambda u_\lambda) = \Sigma_{\gamma,\lambda\in\Gamma} a_\gamma \sigma_\gamma(b_\lambda) u_{\gamma\lambda} \in \ell^1(\Gamma, L^1(X, \mu)).$$

If $x \in \mathcal{H}$ is such that $x \cdot \eta \in \mathcal{H}$ for all $\eta \in \mathcal{H}$, then, by the closed graph theorem, convolution by x describes a bounded operator on $\mathcal{H}$, e.g. if $\gamma \in \Gamma$, then this condition is easily checked for u_γ, and convolution by u_γ gives a unitary operator. We may then consider

$$L^\infty(X, \mu) \rtimes \Gamma = \{x \in \mathcal{H} \mid x \cdot \eta \in \mathcal{H}\ \forall \eta \in \mathcal{H}\} \subset \mathcal{B}(\mathcal{H}).$$

$L^\infty(X, \mu) \rtimes \Gamma$ is a finite von Neumann algebra which contains $L^\infty(X, \mu)$ as a von Neumann subalgebra and has a faithful normal tracial state given by

$$\tau(\Sigma_{\gamma\in\Gamma} a_\gamma u_\gamma) = \int a_e \, d\mu.$$

If (X, μ) is a one-point probability space, then the above construction gives rise to the group von Neumann algebra, which we will denote by $L\Gamma$. Note that, in general, we always have $L\Gamma \subset L^\infty(X, \mu) \rtimes \Gamma$ by considering the sums above for which a_γ is constant for all $\gamma \in \Gamma$.

The connection between the group-measure space construction and orbit equivalence is due to Singer, who showed in [**50**] that two free measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent if and only if there is an isomorphism $\theta : L^\infty(X, \mu) \rtimes \Gamma \to L^\infty(Y, \nu) \rtimes \Lambda$ such that $\theta(L^\infty(X, \mu)) = L^\infty(Y, \nu)$.

The 'representation theory' of a finite von Neumann algebra is captured in the structure of its bimodules, also called correspondences (cf. [**32**]). The theory of correspondences was first developed by Connes [**8**].

Definition 2.2. Let (M, τ) be a finite von Neumann algebra. An *M–M Hilbert bimodule* is a Hilbert space $\mathcal{H}$ equipped with a representation $\pi : M \otimes_{\text{alg}} M^{\text{op}} \to B(\mathcal{H})$ which is normal when restricted to M and M^{op}. We write $\pi(x \otimes y^{\text{op}})\xi$ as $x\xi y$.

An M–M Hilbert bimodule $\mathcal{H}$ is *contained* in an M–M Hilbert bimodule $\mathcal{K}$ if there is a linear isometry $V : \mathcal{H} \to \mathcal{K}$ such that $V(x\xi y) = xV(\xi)y$ for all $\xi \in \mathcal{H}$, $x, y \in M$; $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ if for any $\xi \in \mathcal{H}$, $F \subset M$ finite, and $\varepsilon > 0$, there exist $\xi'_1, \ldots, \xi'_n \in \mathcal{K}$ such that $|\langle x\xi y, \xi\rangle - \Sigma_{k=1}^n \langle x\xi'_k y, \xi'_k\rangle| < \varepsilon$ for all $x, y \in F$. The *trivial bimodule* is the space $L^2(M, \tau)$ with the bimodule structure induced by left and right multiplication; the *coarse bimodule* is the space $L^2(M, \tau) \otimes L^2(M, \tau)$ with the bimodule structure induced by left multiplication on the first factor and right multiplication on the second. The trivial and coarse bimodules play analogous roles in the theory of M–M Hilbert bimodules to the roles played, respectively, by the trivial and left regular representations in the theory of unitary representations of locally compact groups. Note that an M–M correspondence

$\mathcal{H}$ contains the trivial correspondence if and only if $\mathcal{H}$ has non-zero M-central vectors (a vector ξ is M-central if $x\xi = \xi x$ for all $x \in M$).

Given $\xi, \eta \in \mathcal{H}$, note that the maps $M \ni x \mapsto \langle x\xi, \eta\rangle, \langle \xi x, \eta\rangle$ are normal linear functionals on M. A vector $\xi \in \mathcal{H}$ is called *left (respectively, right) bounded* if there exists $C > 0$ such that for every $x \in M$, $\|x\xi\| \leq C\|x\|_2$ (respectively, $\|\xi x\| \leq C\|x\|_2$). The set of vectors which are both left- and right-bounded forms a dense subspace of $\mathcal{H}$. By [**32**], to ξ, a left-bounded vector, we can associate a completely positive map $\phi_\xi : M \to M$ such that for all $x, y \in M$, $\|x\xi y\| = \tau(x^* x \phi_\xi(yy^*))^{1/2}$. If ξ is also right bounded, then this map is seen to naturally extend to a bounded operator $\hat{\phi}_\xi : L^2(M, \tau) \to L^2(M, \tau)$.

Given two M–M Hilbert bimodules $\mathcal{H}$ and $\mathcal{K}$, there is a well-defined tensor product $\mathcal{H} \otimes_M \mathcal{K}$ in the category of M–M Hilbert bimodules: see [**32**] for details.

Definition 2.3. (Compare with Definition 2.1) An M–M Hilbert bimodule is said to:

(1) be *weakly mixing* if $\mathcal{H} \otimes_M \mathcal{H}^{\mathrm{op}}$ does not contain the trivial M–M Hilbert bimodule;

(2) be *mixing* if for every sequence $u_i \in \mathcal{U}(M)$ such that $u_i \to 0$, weakly, we have

$$\lim_{i\to\infty} \sup_{\|x\|\leq 1} \langle u_i \xi x, \eta\rangle = \lim_{i\to\infty} \sup_{\|x\|\leq 1} \langle x\xi u_i, \eta\rangle = 0$$

for all $\xi, \eta \in \mathcal{H}$ (equivalently, $\hat{\phi}_\xi$ is a compact operator from M with the uniform topology to $L^2(M, \tau)$, for every left-bounded vector $\xi \in \mathcal{H}$);

(3) have a *spectral gap* if there exist $x_1, \ldots, x_n \in M$ such that

$$\|\xi - P(\xi)\| \leq \sum_{i=1}^{n} \|x_i\xi - \xi x_i\| \quad \text{for all } \xi \in \mathcal{H},$$

where P is the projection onto the central vectors;

(4) have a *stable spectral gap* if $\mathcal{H} \otimes_M \mathcal{H}^{\mathrm{op}}$ has a spectral gap;

(5) be *amenable* if it is either not weakly mixing or does not have a stable spectral gap.

If $\mathcal{H}$ is a mixing M-correspondence and $\mathcal{K}$ an arbitrary M-correspondence, then $\mathcal{H} \otimes_M \mathcal{K}$ (and also $\mathcal{K} \otimes_M \mathcal{H}$) is mixing, since $\hat{\phi}_{\xi \otimes_M \eta} = \hat{\phi}_\eta \circ \hat{\phi}_\xi$ if ξ and η are both left and right bounded.

Let $\mathcal{H}$ and $\mathcal{K}$ be M–M correspondences, and denote by $\mathcal{H}_0$ and $\mathcal{K}_0$ the sets of right-bounded vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively. For $\xi, \eta \in \mathcal{H}_0$, denote by $(\xi|\eta)$ the element of M such that $\langle \xi x, \eta y\rangle = \tau(y^*(\xi|\eta)x)$ for all $x, y \in M$ (by normality of the map $z \mapsto \langle \xi z, \eta\rangle$, there exists such a $(\xi|\eta) \in L^1(M, \tau)$; right boundedness of ξ and η implies that $(\xi|\eta) \in M$). It is clear that $(\cdot|\cdot)$ is a bilinear map $\mathcal{H}_0 \times \mathcal{H}_0 \to M$ such that $(\xi|\xi) \geq 0$ and $(\xi|\xi) = 0$ if and only if $\xi = 0$ for all $\xi \in \mathcal{H}_0$. For $\xi \in \mathcal{H}_0$ and $\eta \in \mathcal{K}_0$, define the linear map $T_{\xi,\eta} : \mathcal{H}_0 \to \mathcal{K}_0^{\mathrm{op}}$ by $T_{\xi,\eta}(\cdot) = (\cdot|\xi)\eta^{\mathrm{op}}$. It is not hard to check that $T_{\xi,\eta}$ is bounded with $\|T_{\xi,\eta}\| \leq \|(\xi|\xi)\|\|(\eta|\eta)\|$; hence, $T_{\xi,\eta}$ extends to a bounded operator $\mathcal{H} \to \mathcal{K}^{\mathrm{op}}$. Let $\mathcal{L}^2_M(\mathcal{H}, \mathcal{K})$ be the subspace of $B(\mathcal{H}, \mathcal{K}^{\mathrm{op}})$ which is the closed span of all operators of the form $T_{\xi,\eta}$ under the Hilbert norm $\|T_{\xi,\eta}\|_{\mathcal{L}^2_M} = \tau((\xi|\xi)(\eta|\eta))^{1/2}$. Moreover, $\mathcal{L}^2_M(\mathcal{H}, \mathcal{K})$ is equipped with a natural M–M Hilbert bimodule structure given by $(x \otimes y^{\mathrm{op}})(T_{\xi,\eta}) = T_{x\xi, y\eta}$ identifying it with $\mathcal{H} \otimes_M \mathcal{K}^{\mathrm{op}}$. Note that if $T \in \mathcal{L}^2_M(\mathcal{H}, \mathcal{K})$, then $(T^*T)^{1/2} \in \mathcal{L}^2_M(\mathcal{H}, \mathcal{H})$.

PROPOSITION 2.4. *An M–M correspondence $\mathcal{H}$ is weakly mixing if and only if for any M–M correspondence $\mathcal{K}$, $\mathcal{H} \otimes_M \mathcal{K}^{\mathrm{op}}$ does not contain the trivial correspondence.*

Proof. The reverse implication is trivial. Conversely, suppose that there exists $\mathcal{K}$ such that $\mathcal{H} \otimes \mathcal{K}^{\mathrm{op}}$ contains an M-central vector. Identifying $\mathcal{H} \otimes_M \mathcal{K}^{\mathrm{op}}$ with $\mathcal{L}^2_M(\mathcal{H}, \mathcal{K})$, let $T \in \mathcal{L}^2_M(\mathcal{H}, \mathcal{K})$ be an M-central vector. Then $(T^*T)^{1/2} \in \mathcal{L}^2_M(\mathcal{H}, \mathcal{H})$ is an M-central vector; hence, $\mathcal{H}$ is not weakly mixing. □

2.1. *Gaussian actions.* Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of a countable discrete group Γ. The aim of this section is to describe the construction of a measure-preserving action of Γ on a non-atomic standard probability space (X, μ) such that $\mathcal{H}$ is realized as a subspace of $L^2_{\mathbb{R}}(X, \mu)$ and π is contained in the Koopman representation $\Gamma \curvearrowright L^2_0(X, \mu)$. The action $\Gamma \curvearrowright (X, \mu)$ is referred to as the *Gaussian action* associated to π. We give an operator-algebraic alternative construction of the Gaussian action similar to Voiculescu's construction of free semicircular random variables.

Given a real Hilbert space $\mathcal{H}$, the *n-symmetric tensor* $\mathcal{H}^{\odot n}$ is the subspace of $\mathcal{H}^{\otimes n}$ fixed by the action of the symmetric group S_n by permuting the indices. For $\xi_1, \ldots, \xi_n \in \mathcal{H}$, we define their symmetric tensor product $\xi_1 \odot \cdots \odot \xi_n \in \mathcal{H}^{\odot n}$ to be $(1/n!) \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$. Denote

$$\mathfrak{S}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} (\mathcal{H} \otimes \mathbb{C})^{\odot n},$$

with Ω the vacuum vector and having renormalized inner product such that $\|\xi\|^2_{\mathfrak{S}(\mathcal{H})} = n!\|\xi\|^2$ for $\xi \in \mathcal{H}^{\odot n}$.

For $\xi \in \mathcal{H}$, let x_ξ be the *symmetric creation operator*,

$$x_\xi(\Omega) = \xi, \quad x_\xi(\eta_1 \odot \cdots \odot \eta_k) = \xi \odot \eta_1 \odot \cdots \odot \eta_k,$$

and its adjoint, $\partial/\partial\xi = (x_\xi)^*$

$$\frac{\partial}{\partial\xi}(\Omega) = 0, \quad \frac{\partial}{\partial\xi}(\eta_1 \odot \cdots \odot \eta_k) = \sum_{i=1}^{k} \langle \xi, \eta_i \rangle \eta_1 \odot \cdots \odot \widehat{\eta_i} \odot \cdots \odot \eta_k.$$

Let

$$s(\xi) = \frac{1}{2}\left(x_\xi + \frac{\partial}{\partial\xi}\right),$$

and note that it is an unbounded, self-adjoint operator on $\mathfrak{S}(\mathcal{H})$.

The *moment generating function* $M(t)$ for the Gaussian distribution is defined to be

$$M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(tx) \exp(-x^2/2)\, dx = \exp(t^2/2).$$

It is easy to check that if $\|\xi\| = 1$, then

$$\langle s(\xi)^n \Omega, \Omega \rangle = M^{(n)}(0) = \frac{(2k)!}{2^k k!}$$

if $n = 2k$ and 0 if n is odd. Hence, $s(\xi)$ may be regarded as a Gaussian random variable. Note that if $\xi, \eta \in \mathcal{H}$, then $s(\xi)$ and $s(\eta)$ commute; moreover, if $\xi \perp \eta$, then

$$\langle s(\xi)^m s(\eta)^n \Omega, \Omega \rangle = \langle s(\xi)^m \Omega, \Omega \rangle \langle s(\eta)^n \Omega, \Omega \rangle \quad \text{for all } m, n \in \mathbb{N};$$

thus, $s(\xi)$ and $s(\eta)$ are independent random variables.

From now on, we will use the convention $\xi_1\xi_2\cdots\xi_k$ to denote the symmetric tensor $\xi_1 \odot \xi_2 \odot \cdots \odot \xi_k$. Let Ξ be a basis for $\mathcal{H}$ and

$$\mathcal{S}(\Xi) = \{\Omega\} \cup \{s(\xi_1)s(\xi_2)\cdots s(\xi_k)\Omega : \xi_1, \xi_2, \ldots, \xi_k \in \Xi\}.$$

LEMMA 2.5. *The set $\mathcal{S}(\Xi)$ is a (non-orthonormal) basis of $\mathfrak{S}(\mathcal{H})$.*

Proof. We will show that $\xi_1\cdots\xi_k \in \mathrm{span}(\mathcal{S}(\Xi))$ for all $\xi_1, \ldots, \xi_k \in \mathcal{H}$. We have $\Omega \in \mathrm{span}(\mathcal{S}(\Xi))$. Also, since $s(\xi)\Omega = \xi$, $\mathcal{H} \subset \mathrm{span}(\mathcal{S}(\Xi))$. Now, as $s(\xi_1)\cdots s(\xi_k)\Omega = P(\xi_1, \ldots, \xi_k)$ is a polynomial in $\xi_1, \ldots, \xi_k$ of degree k with top term $\xi_1\cdots\xi_k$, the result follows by induction on k. □

Let

$$u(\xi_1, \ldots, \xi_k) = \exp(\pi i s(\xi_1)\cdots s(\xi_k)) \quad \text{and} \quad u(\xi_1, \ldots, \xi_k)^t = \exp(\pi i t s(\xi_1)\cdots s(\xi_k)).$$

Denote by A the von Neumann algebra generated by all such $u(\xi_1, \ldots, \xi_k)$, which is the same as the von Neumann algebra generated by the spectral projections of the unbounded operators $s(\xi_1)\cdots s(\xi_k)$.

THEOREM 2.6. *We have $L^2(A, \tau) \cong \mathfrak{S}(\mathcal{H})$, and A is a maximal abelian $*$-subalgebra of $B(\mathfrak{S}(\mathcal{H}))$ with faithful trace $\tau = \langle \cdot\, \Omega, \Omega\rangle$. In particular, A is a diffuse abelian von Neumann algebra.*

Proof. By Lemma 2.5, $A \mapsto A\Omega$ is an embedding of A into $\mathfrak{S}(\mathcal{H})$. By Stone's theorem,

$$\lim_{t\to 0} \frac{u(\xi_1, \ldots, \xi_k)^t - 1}{\pi i t}\Omega = s(\xi_1)\cdots s(\xi_k)\Omega;$$

hence, $A\Omega$ is dense in $\mathfrak{S}(\mathcal{H})$. This implies that A is maximal abelian in $B(\mathfrak{S}(\mathcal{H}))$. □

There is a natural strongly continuous embedding $\mathcal{O}(\mathcal{H}) \hookrightarrow \mathcal{U}(\mathfrak{S}(\mathcal{H}))$ given by

$$T \mapsto T^{\mathfrak{S}} = 1 \oplus \bigoplus_{n=1}^{\infty} T^{\odot n}.$$

It follows that there is an embedding $\mathcal{O}(\mathcal{H}) \to \mathrm{Aut}(A, \tau)$, $T \mapsto \sigma_T$, which can be identified on the unitaries $u(\xi_1, \ldots, \xi_k)$ by

$$\sigma_T(u(\xi_1, \ldots, \xi_k)) = \mathrm{Ad}(T^{\mathfrak{S}})(u(\xi_1, \ldots, \xi_k)) = u(T(\xi_1), \ldots, T(\xi_k)).$$

Thus, for an orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, there is a natural action $\sigma^\pi : \Gamma \to \mathrm{Aut}(A, \tau)$ given by

$$\sigma^\pi_\gamma(u(\xi_1, \ldots, \xi_k)) = u(\pi_\gamma(\xi_1), \ldots, \pi_\gamma(\xi_k)) = \mathrm{Ad}(\pi_\gamma^{\mathfrak{S}})(u(\xi_1, \ldots, \xi_k)).$$

The action σ^π is the *Gaussian action* associated to π.

We have that ergodic properties which remain stable with respect to tensor products transfer from π to σ^π.

PROPOSITION 2.7. *In particular, for a subgroup $H \leq \Gamma$, $\sigma^\pi|_H$ possesses any of the following properties if and only if $\pi|_H$ does:*

(1) *weakly mixing;*
(2) *mixing;*
(3) *stable spectral gap;*
(4) *being contained in a direct sum of copies of the left regular representation;*
(5) *being weakly contained in the left regular representation.*

For Gaussian actions, stable properties are equivalent to their 'non-stable' counterparts. The following proposition serves as a prototype of such a result, showing that ergodicity implies stable ergodicity, i.e. weakly mixing.

THEOREM 2.8. *$\Gamma \curvearrowright^{\sigma^\pi} (A, \tau)$ is ergodic if and only if π is weakly mixing.*

Proof. The reverse implication follows from Proposition 2.7. Conversely, suppose that there exists $\xi \in \mathcal{H}^{\otimes 2}$ such that for all $\gamma \in \Gamma$, $\pi_\gamma^2(\xi) = \xi$. Viewing ξ as a Hilbert–Schmidt operator on $\mathcal{H}$, let $|\xi| = (\xi\xi^*)^{1/2}$. Since the map $\xi \otimes \eta \mapsto \eta \otimes \xi$ is the same as taking the adjoint of the corresponding Hilbert–Schmidt operator, we have $|\xi| \in \mathcal{H}^{\odot 2}$ and $\pi_\gamma(|\xi|) = |\xi|$. By functional calculus, there exists $\lambda > 0$ such that $\eta = E_\lambda(|\xi|) \neq 0$ is a finite rank operator. Hence, $\eta = \eta_{11} \odot \eta_{12} + \cdots + \eta_{n1} \odot \eta_{n2} \in \mathcal{H}^{\odot 2}$ with $\eta_{i1} \odot \eta_{i2} \perp \eta_{j1} \odot \eta_{j2}$ for $i \neq j$. But then $u = \prod_{i=1}^n u(\eta_{i1}, \eta_{i2}) \in A$, a non-trivial unitary and $\sigma_\gamma^\pi(u) = u$. Hence, σ^π is not ergodic. □

2.2. *Cocycles from representations and from actions.* Let $\mathcal{K}$ be a real Hilbert space and $\pi : \Gamma \to \mathcal{O}(\mathcal{K})$ an orthogonal representation of a countable discrete group Γ.

Definition 2.9. A 1-*cocycle* is a map $b : \Gamma \to \mathcal{K}$ satisfying the cocycle identity $b(\gamma_1\gamma_2) = \pi_{\gamma_1} b(\gamma_2) + b(\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$. A 1-cocycle is a *coboundary* is there exists $\eta \in \mathcal{K}$ such that $b(\gamma) = \pi_\gamma \eta - \eta$ for all $\gamma \in \Gamma$.

It is a well-known fact (cf. [**2**]) that a 1-cocycle b is a coboundary if and only if $\sup_{\gamma\in\Gamma} \|b(\gamma)\| < \infty$. Let $Z^1(\Gamma, \pi)$ and $B^1(\Gamma, \pi)$ denote, respectively, the vector space of all 1-cocycles and the subspace of coboundaries. The *first cohomology space* $H^1(\Gamma, \pi)$ of the representation π is then defined to be $Z^1(\Gamma, \pi)/B^1(\Gamma, \pi)$.

Let $\Gamma \curvearrowright^\sigma (X, \mu)$ be an ergodic, measure-preserving action on a standard probability space (X, μ) and let $\mathbb{A}$ be a Polish topological group.

Definition 2.10. A 1-*cocycle* is a measurable map $c : \Gamma \times X \to \mathbb{A}$ satisfying the cocycle identity $c(\gamma_1\gamma_2, x) = c(\gamma_1, \sigma_{\gamma_2}(x))c(\gamma_2, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and almost every $x \in X$. A pair of 1-cocycles c_1, c_2 are *cohomologous* (written $c_1 \sim c_2$) if there exists a measurable map $\xi : X \to \mathbb{A}$ such that $\xi(\sigma_\gamma(x))c_1(\gamma, x)\xi(x)^{-1} = c_2(\gamma, x)$ for all $\gamma \in \Gamma$ and almost every $x \in X$. A 1-cocycle is a *coboundary* if it is cohomologous to the cocycle which is identically 1.

Let $Z^1(\Gamma, \sigma, \mathbb{A})$ and $B^1(\Gamma, \sigma, \mathbb{A})$ denote, respectively, the space of all 1-cocycles and the subspace of coboundaries. The *first cohomology space* $H^1(\Gamma, \sigma, \mathbb{A})$ of the action σ is defined to be $Z^1(\Gamma, \sigma, \mathbb{A})/\sim$. Note that if $\mathbb{A}$ is abelian, $Z^1(\Gamma, \sigma, \mathbb{A})$ is endowed with a natural abelian group structure and $H^1(\Gamma, \sigma, \mathbb{A}) = Z^1(\Gamma, \sigma, \mathbb{A})/B^1(\Gamma, \sigma, \mathbb{A})$. To any homomorphism $\rho : \Gamma \to \mathbb{A}$ we can associate a cocycle $\tilde{\rho}$ by $\tilde{\rho}(\gamma, x) = \rho(\gamma)$. Using terminology developed by Popa (cf. [**38**]), a 1-cocycle c is said to *untwist* if there exists a homomorphism $\rho : \Gamma \to \mathbb{A}$ such that c is cohomologous to $\tilde{\rho}$. To any 1-cocycle $c \in Z^1(\Gamma, \sigma, \mathbb{A})$, we can associate two 1-cocycles $c_\ell, c_r \in Z^1(\Gamma, \sigma \times \sigma, \mathbb{A})$ given by $c_\ell(\gamma, (x, y)) = c(\gamma, x)$ and $c_r(\gamma, (x, y)) = c(\gamma, y)$. It is easy to check that c untwists only if c_ℓ is cohomologous to c_r; if σ is weakly mixing, [**38**, Theorem 3.1] establishes the converse.
Note that, for brevity, we will drop the '1' when discussing 1-cocycles of representations or actions.

The pertinence of the 1-cohomology of group actions to ergodic theory is that it provides a natural, and rather powerful, technical framework for the orbit equivalence theory of free ergodic actions of countable discrete groups. We give a brief account of this connection: details may found in, for instance, [**16, 55**].

Consider two free, ergodic, measure-preserving actions $\Gamma \curvearrowright^{\sigma} (X, \mu)$ and $\Lambda \curvearrowright^{\rho} (Y, \nu)$ of countable discrete groups Γ and Λ on respective standard probability spaces (X, μ) and (Y, ν).

Definition 2.11. The actions $\Gamma \curvearrowright^{\sigma} (X, \mu)$ and $\Lambda \curvearrowright^{\rho} (Y, \nu)$ are *orbit equivalent* if there exists a probability space isomorphism $\Phi : X \to Y$ such that $\Phi(\Gamma x) = \Lambda\, \Phi(x)$ for almost every $x \in X$. The actions are *conjugate* if there exists an isomorphism of groups $\phi : \Gamma \to \Lambda$ and a probability space isomorphism $\Phi : X \to Y$ such that $\Phi(\gamma x) = \phi(\gamma)\Phi(x)$ for all $\gamma \in \Gamma$ and almost every $x \in X$.

It is clear that orbit equivalence is weaker than conjugacy. Given an orbit equivalence Φ from $\Gamma \curvearrowright (X, \mu)$ to $\Lambda \curvearrowright (Y, \nu)$, we would like to describe how far Φ departs from implementing a conjugacy. Since the actions are free, for almost every $x \in X$, for every $\gamma \in \Gamma$, there exists a unique $\lambda \in \Lambda$ such that $\Phi(\gamma x) = \lambda\Phi(x)$. One can easily verify that the map $c : \Gamma \times X \to \Lambda$ which selects the λ corresponding to the pair (γ, x) is almost everywhere well defined and measurable. From the fact that Φ preserves orbits, it follows that c is a cocycle, the *Zimmer cocycle*, associated to Φ. It is a classical result that, accounting for finite normal subgroups $H < \Gamma$ and $K < \Lambda$, the Zimmer cocycle c will untwist if and only if Φ is implemented by a conjugacy; precisely, there are an isomorphism of groups $\psi : \Gamma/H \to \Lambda/K$ and a probability space isomorphism $\Psi : X \to Y$ such that $\Phi(\gamma x) \in \psi(\gamma H)K\Psi(x)$ for all $\gamma \in G$ and for almost every $x \in X$, cf. [**55**].

This strategy of conceptualizing orbit equivalence theory in the broader context of cohomology is particularly useful when one wants to show that some type of orbit equivalence *rigidity* holds for an action $\Gamma \curvearrowright^{\sigma} (X, \mu)$; that is, given some 'nice' class of group actions $\mathcal{L}$, of which, say, $\Lambda \curvearrowright (Y, \nu)$ is a representative, any orbit equivalence between $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ is implemented by a conjugacy (of course, excepting finite normal subgroups). To do so, it is sufficient to demonstrate that the action is *superrigid*, meaning that rigidity is dependent only on the target group Λ and not the action $\Lambda \curvearrowright (Y, \nu)$. In practice, this amounts to showing that every cocycle $c \in Z^1(\Gamma, \sigma, \Lambda)$ untwists.

2.3. *Closable derivations.* We review here briefly some general properties of closable derivations on a finite von Neumann algebra and set up some notation to be used in the following. For a more detailed discussion, see [**12, 28, 29**] or [**26**].

Definition 2.12. Let (N, τ) be a finite von Neumann algebra and $\mathcal{H}$ an N–N correspondence. A *derivation* δ is an unbounded operator $\delta : L^2(N, \tau) \to \mathcal{H}$ such that $D(\delta)$ is a $\|\cdot\|_2$-dense $*$-subalgebra of N, and $\delta(xy) = x\delta(y) + \delta(x)y$, for each $x, y \in D(\delta)$. A derivation is *closable* if it is closable as an operator and *real* if $\mathcal{H}$ has an antilinear involution $\mathcal{J}$ such that $\mathcal{J}(x\xi y) = y^*\mathcal{J}(\xi)x^*$, and $\mathcal{J}(\delta(z)) = \delta(z^*)$, for each $x, y \in N$, $\xi \in \mathcal{H}$, $z \in D(\delta)$.

If δ is a closable derivation, then, by [**12**], $D(\overline{\delta}) \cap N$ is again a $*$-subalgebra and $\delta_{|D(\overline{\delta}\cap N)}$ is again a derivation. We will thus use the slight abuse of notation by saying that $\overline{\delta}$ is a closed derivation.

To every closed real derivation $\delta : N \to \mathcal{H}$, we can associate a semigroup deformation $\Phi^t = \exp(-t\delta^*\overline{\delta})$, $t > 0$, and a resolvent deformation $\zeta_\alpha = (\alpha/(\alpha + \delta^*\overline{\delta}))^{1/2}$, $\alpha > 0$. Both of these deformations are of unital, symmetric, completely positive maps; moreover, the derivation δ can be recovered from these deformations [**45, 46**].

We also have that the deformation Φ^t converges uniformly on $(N)_1$ as $t \to 0$ if and only if the deformation ζ_α converges uniformly on $(N)_1$ as $\alpha \to \infty$.

Definition 2.13. [**29**, Definition 4.1] Let (N, τ) be a finite von Neumann algebra. N is L^2-*rigid* if given any inclusion $(N, \tau) \subset (M, \tilde{\tau})$, and any closable real derivation $\delta : M \to \mathcal{H}$ such that $\mathcal{H}$ when viewed as an N–N correspondence embeds in $(L^2N \overline{\otimes} L^2N)^{\oplus\infty}$, we then have that the associated deformation ζ_α converges uniformly to the identity in $\|\cdot\|_2$ on the unit ball of N.

We point out here that our definition above is formally stronger than the one given in [**29**]. Specifically, there it was assumed that $\mathcal{H}$ embedded into the coarse bimodule as an M–M bimodule rather than an N–N bimodule. However, this extra condition was not used in [**29**], and since the above definition has better stability properties (see Theorem 6.3) we have chosen to use the same terminology.

Examples of non-amenable groups which do not give rise to L^2-rigid group von Neumann algebras are groups such that the first ℓ^2-Betti number is positive. These are, in fact, the only known examples, and L^2-rigidity should be viewed as a von Neumann analog of vanishing first ℓ^2-Betti number.

Showing that a group von Neumann algebra is L^2-rigid can be quite difficult in general, since one has to consider derivations which may not be defined on the group algebra. Nonetheless, there are certain situations where this can be verified.

THEOREM 2.14. [**29**, Corollary 4.6] *Let Γ be a non-amenable countable discrete group. If $L\Gamma$ is weakly rigid, non-prime, or has property* (Γ) *of Murray and von Neumann, then $L\Gamma$ is L^2-rigid.*

We give another class of examples below (see also [**25, 26**] or [**30**]). The gap between group von Neumann algebras which are known to be L^2-rigid and groups with vanishing first ℓ^2-Betti number is, however, quite large. For example, as we mentioned in the introduction, the wreath product $\mathbb{Z} \wr \mathbb{F}_2$ is a group which has vanishing first ℓ^2-Betti number but for which it is not known whether the group von Neumann algebra is L^2-rigid.

3. *Deformations*

In this section and §5, we will discuss the interplay between one-parameter groups of automorphisms or, more generally, semigroups of completely positive maps of finite factors (deformations) and their infinitesimal generators (derivations). The motivation for studying deformations at the infinitesimal level is that it allows for the creation of other related deformations of the algebra. And, while Popa's deformation/rigidity machinery requires uniform convergence of the original deformation on some target subalgebra, it is often more feasible to demonstrate uniform convergence of a related deformation and then transfer those estimates back to the original.

We begin by recalling Popa's notion of an s-malleable deformation, and give some examples of such deformations that have appeared in the literature.

Definition 3.1. [**38**, Definition 4.3] Let (M, τ) be a finite von Neumann algebra such that $(M, \tau) \subset (\tilde{M}, \tilde{\tau})$, where $(\tilde{M}, \tilde{\tau})$ is another finite von Neumann algebra. A pair (α, β), consisting of a point-wise strongly continuous one-parameter family $\alpha : \mathbb{R} \to \mathrm{Aut}(\tilde{M}, \tilde{\tau})$ and an involution $\beta \in \mathrm{Aut}(\tilde{M}, \tilde{\tau})$, is called an *s-malleable deformation* of M if:

(1) $M \subset \tilde{M}^{\beta}$;

(2) $\alpha_t \circ \beta = \beta \circ \alpha_{-t}$; and

(3) $\alpha_1(M) \perp M$.

3.1. *Popa's deformation.* The following deformation was used by Popa in [**38**] to obtain cocycle superrigidity for generalized Bernoulli actions of property (T) groups.

Let (A, τ) be a finite diffuse abelian von Neumann algebra and $u, v \in A \otimes A$ be generating Haar unitaries for $A \otimes 1$, $1 \otimes A \subset A \otimes A$, respectively. Set $w = u^* v$. Choose $h \in A \otimes A$ self-adjoint such that $\exp(\pi i h) = w$, and let $w^t = \exp(\pi i t h)$. Since $\{w\}'' \perp A \otimes 1, 1 \otimes A$, we have that for any t, $w^t u$ and $w^t v$ are again Haar unitaries. Moreover, since $w \in \{w^t u, w^t v\}''$, $\{w^t u, w^t v\}$ is a pair of generating Haar unitaries in $A \otimes A$. Hence, there is a well-defined one-parameter family $\alpha : \mathbb{R} \to \mathrm{Aut}(A \otimes A, \tau \otimes \tau)$ given by

$$\alpha_t(u) = w^t u, \quad \alpha_t(v) = w^t v.$$

The family α, together with the automorphism β given by

$$\beta(u) = u, \quad \beta(v) = u^2 v^*,$$

is seen to be an s-malleable deformation of $A \otimes 1 \subset A \otimes A$.

Definition 3.2. Let (P, τ) be a finite von Neumann algebra and $\sigma : \Gamma \to \mathrm{Aut}(P, \tau)$ a Γ-action. $\Gamma \curvearrowright^{\sigma} (P, \tau)$ is an *s-malleable action* if there exists an s-malleable deformation (α, β) of (P, τ) such that β and α_t commute with $\sigma_\gamma \otimes \sigma_\gamma$ for all $t \in \mathbb{R}$, $\gamma \in \Gamma$.

For any countable discrete group there is a canonical example of an s-malleable action, the *Bernoulli shift*. Let $(A, \tau) = (L^\infty(\mathbb{T}, \lambda), \int \cdot \, d\lambda)$, $(X, \mu) = \prod_{g \in \Gamma} (\mathbb{T}, \lambda)$, and $(B, \tau') = \bigotimes_{\gamma \in \Gamma} (A, \tau)$. The Bernoulli shift is the natural action $\Gamma \curvearrowright^{\sigma} (X, \mu)$ defined by shifting indices: $\sigma_{\gamma_0}((x_\gamma)_\gamma) = (x_\gamma)_{\gamma_0 \gamma} = (x_{\gamma_0^{-1}\gamma})_\gamma$. Defining

$$\tilde{\alpha}_t((\tilde{x}_\gamma)_\gamma) = (\alpha_t(\tilde{x}_\gamma))_\gamma$$

and

$$\tilde{\beta}((\tilde{x}_\gamma)_\gamma) = (\beta(\tilde{x}_\gamma))_\gamma$$

for $(\tilde{x}_\gamma)_\gamma \in \tilde{B} = \bigotimes_{\gamma \in \Gamma} (A \otimes A) \cong B \otimes B$, we see that $(\tilde{\alpha}, \tilde{\beta})$ is an s-malleable deformation of B which commutes with the Bernoulli Γ-action.

3.2. *Ioana's deformation.* The deformation described below was first used by Ioana [**19**] in the case when the base space is non-amenable, and later used by Chifan and Ioana [**6**] in part to obtain solidity of $L^\infty(X, \mu) \rtimes_\sigma \Gamma$, whenever $L\Gamma$ is solid and $\Gamma \curvearrowright^\sigma (X, \mu)$ is the Bernoulli shift. (A finite von Neumann algebra M is solid if $B' \cap M$ is amenable whenever $B \subset M$ does not have minimal projections.) Their deformation was inspired by the free product deformation used in [**20**]. A similar deformation has also been previously used by Voiculescu in [**54**].

Given a finite von Neumann algebra (B, τ), let $\tilde{B} = B * L\mathbb{Z}$, the free product of the von Neumann algebras B and $L\mathbb{Z}$. If $u \in \mathcal{U}(L\mathbb{Z})$ is a generating Haar unitary, choose an $h \in L\mathbb{Z}$ such that $\exp(\pi i h) = u$ and let $u^t = \exp(\pi i t h)$. Define the deformation $\alpha : \mathbb{R} \to \mathrm{Aut}(\tilde{B}, \tilde{\tau})$ by

$$\alpha_t = \mathrm{Ad}(u^t).$$

Let $\beta \in \mathrm{Aut}(\tilde{B}, \tilde{\tau})$ be defined by

$$\beta_{|B} = \mathrm{id} \quad \text{and} \quad \beta(u) = u^*.$$

It is easy to check that (α, β) is an s-malleable deformation of B.

If a countable discrete group Γ acts on a countable set S, then we may consider the *generalized Bernoulli shift action* of Γ on $\bigotimes_{s \in S} B$ given by $\sigma_\gamma(\bigotimes_{s \in S} b_s) = \bigotimes_{s \in S} b_{\gamma^{-1}s}$. We then have $\bigotimes_{s \in S} B \subset \bigotimes_{s \in S} \tilde{B}$ and $(\bigotimes_{s \in S} \alpha, \bigotimes_{s \in S} \beta)$ gives an s-malleable deformation of $\bigotimes_{s \in S} B$.

3.3. *Malleable deformations of Gaussian actions.* We will now construct the canonical s-malleable deformation of a Gaussian action which is given in [**15**, §4.3] and give an explicit description of its associated derivation. To begin, let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation, $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, and $\tilde{\pi} = \pi \oplus \pi$. If $\sigma^\pi : \Gamma \to \mathrm{Aut}(A, \tau)$ is the Gaussian action associated with π, then the Gaussian action associated to $\tilde{\pi}$ is naturally identified with the action $\sigma^\pi \otimes \sigma^\pi$ on $A \otimes A$. Let $\tilde{\sigma}^\pi = \sigma^\pi \otimes \sigma^\pi$.

Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the operator which gives $\tilde{\mathcal{H}}$ the structure of a complex Hilbert space, and consider the one-parameter unitary group $\theta_t = \exp((\pi t/2)J)$. Since θ_t commutes with $\tilde{\pi}$, there is a well-defined one-parameter group $\alpha : \mathbb{R} \to \mathrm{Aut}(A \otimes A, \tau \otimes \tau)$ which commutes with $\tilde{\sigma}^\pi$, namely

$$\alpha_t = \sigma_{\theta_t} = \mathrm{Ad}\left(\exp\left(\frac{\pi t}{2} J\right)^{\mathfrak{S}}\right).$$

Let $\rho = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and observe that $\rho \circ \theta_{-t} = \theta_t \circ \rho$. Hence,

$$\beta = \sigma_\rho = \mathrm{Ad}(\rho^{\mathfrak{S}})$$

conjugates α_t and α_{-t}. Finally, notice that $\theta_1(\mathcal{H} \oplus 0) = 0 \oplus \mathcal{H}$, which gives $\alpha_1(A \otimes 1) = 1 \otimes A$. The pair (α, β) is, thus, an s-malleable deformation of the action σ^π.

Let $T \in B(\tilde{\mathcal{H}})$ be skew adjoint. Associate to T the unbounded skew-adjoint operator $\partial(T)$ on $\mathfrak{S}(\mathcal{H})$ defined by

$$\partial(T)(\Omega) = 0, \quad \partial(T)(\xi_1 \cdots \xi_n) = \sum_{i=1}^{n} \xi_1 \cdots T(\xi_i) \cdots \xi_n.$$

We have that if $U(t) = \exp(tT) \in \mathcal{O}(\mathcal{H})$, then

$$\lim_{t\to 0} \frac{U(t)^{\mathfrak{S}} - I}{t} = \partial(T).$$

Let $\delta : A \otimes A \to L^2(A \otimes A)$ be the derivation defined by

$$\delta(x) = [x, \partial(T)] = \lim_{t\to 0} \frac{\sigma_{U(t)}(x) - x}{t}.$$

Taking T to be the operator J defined above gives us the derivation which is the infinitesimal generator of the s-malleable deformation of the Gaussian action described in this section. From the relation $\delta(\cdot) = [\,\cdot, \partial(J)]$, we see that the $*$-algebra generated by the operators $s(\xi)$ forms a core for δ.

Letting $\delta_0 = \delta|_{A\otimes 1}$, we have

$$\Phi^t = \exp(-t\delta_0^*\overline{\delta}_0) = \exp(-tE_{A\otimes 1} \circ \delta^*\overline{\delta}) = \exp(tE_{A\otimes 1} \circ \delta^2).$$

We compute

$$E_{A\otimes 1} \circ \delta^2(s(\xi_1) \cdots s(\xi_k)) = -ks(\xi_1) \cdots s(\xi_k).$$

Hence,

$$\Phi^t(s(\xi_1) \cdots s(\xi_k)) = (1 - e^{-kt})s(\Omega) + e^{-kt}s(\xi_1) \cdots s(\xi_k).$$

4. *Cohomology of Gaussian actions*

In this section, we obtain Theorem 1.1 and its corollary. We do so by using a construction (cf. [**18, 27, 48**]), which, given an orthogonal representation and a cocycle, produces a $\mathbb{T}$-valued cocycle for the associated Gaussian action. We then show that these cocycles do not untwist by applying the above deformation.

Let $b : \Gamma \to \mathcal{H}$ be a cocycle for an orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ and $\Gamma \curvearrowright^{\sigma} (A, \tau) = (L^\infty(X, \mu), \int \cdot\, d\mu)$ be the Gaussian action associated to π, as described in §2.1. Viewing $\mathcal{H}$ as a subset of $L^2_{\mathbb{R}}(X, \mu)$, Parthasarathy and Schmidt [**27**] constructed the cocycle $c : \Gamma \times X \to \mathbb{T}$ by the rule

$$c(\gamma, x) = \exp(ib(\gamma^{-1}))(x).$$

We write ω_γ for the element of $\mathcal{U}(L^\infty(X, \mu))$ given by $\omega_\gamma(x) = c(\gamma, \gamma^{-1}x)$. The cocycle identity for c then transforms to the formula $\omega_{\gamma_1\gamma_2} = \omega_{\gamma_1}\sigma_{\gamma_1}(\omega_{\gamma_2})$ for all $\gamma_1, \gamma_2 \in \Gamma$. Moreover, c is cohomologous to a homomorphism if and only if there is a unitary element $u \in \mathcal{U}(L^\infty(X, \mu))$ such that $\gamma \mapsto u\omega_\gamma\sigma_\gamma(u^*)$ is a homomorphism, i.e. each $u\omega_\gamma\sigma_\gamma(u^*)$ is fixed by the action of the group.

A routine calculation shows that $\tau(\omega_\gamma) = \int c(\gamma, x)\, d\mu(x) = \exp(-\|b(\gamma)\|^2/2)$. In particular, this shows that the representation associated to the positive-definite function $\varphi(\gamma) = \exp(-\|b(\gamma)\|^2/2)$ is naturally isomorphic to the twisted Gaussian action $\omega_\gamma\sigma_\gamma$.

THEOREM 4.1. *Using the notation above, if $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is weakly mixing (so that σ is ergodic) and if b is an unbounded cocycle, then c does not untwist.*

Proof. Since σ is ergodic, if c were to untwist then there would exist some $u \in \mathcal{U}(A)$ such that $u\omega_\gamma\sigma_\gamma(u) \in \mathbb{T}$ for all $\gamma \in \Gamma$. It would then follow that any deformation of A which commutes with the action of Γ must converge uniformly on the set $\{\omega_\gamma \mid \gamma \in \Gamma\}$. Indeed, this is just a consequence of the fact that completely positive deformations become asymptotically A-bimodular.

However, if we apply the deformation α_t from §3.3, then we can compute

$$\begin{aligned}&\langle \alpha_{2t/\pi}(\omega_\gamma \otimes 1), \omega_\gamma \otimes 1\rangle\\ &\quad = \langle \exp(i(\cos t)b(\gamma^{-1})) \otimes \exp(-i(\sin t)b(\gamma^{-1})), \exp(ib(\gamma^{-1})) \otimes 1\rangle\\ &\quad = \exp((1-\cos t)^2\|b(\gamma)\|^2/2 + (\sin^2 t)\|b(\gamma)\|^2/2)\\ &\quad = \exp(-(1-\cos t)\|b(\gamma)\|^2).\end{aligned}$$

This will converge uniformly for $\gamma \in \Gamma$ if and only if the cocycle b is bounded and hence the result follows. □

COROLLARY 4.2. *The exponentiation map described above induces an injective homomorphism* $H^1(\Gamma, \pi) \to H^1(\Gamma, \sigma, \mathbb{T})/\chi(\Gamma)$*, where* $\chi(\Gamma)$ *is the character group of* Γ.

Proof. It is easy to see that if two cocycles in $Z^1(\Gamma, \pi)$ are cohomologous, then the resulting cocycles for the Gaussian action will also be cohomologous. This shows that the map described above is well defined.

The above theorem, together with the fact that this map is a homomorphism, shows that this map is injective. □

Since a non-amenable group has vanishing first ℓ^2-Betti number if and only if it has vanishing first cohomology into its left regular representation [**3, 31**], we derive the following corollary.

COROLLARY 4.3. *Let* Γ *be a non-amenable countable discrete group and let* $\Gamma \curvearrowright^\sigma (X, \mu)$ *be the Bernoulli shift action. If* $\beta_1^{(2)}(\Gamma) \neq 0$*, then* $H^1(\Gamma, \sigma, \mathbb{T}) \neq \chi(\Gamma)$*, where* $\chi(\Gamma)$ *is the group of characters. In particular,* $\Gamma \curvearrowright^\sigma (X, \mu)$ *is not* $\mathcal{U}_{\text{fin}}$*-cocycle superrigid.*

5. *Derivations*

In this section, we continue our investigation of deformations, but this time on the infinitesimal level.

5.1. *Derivations from s-malleable deformations.*

Let (M, τ) be a finite von Neumann algebra and let $\alpha : \mathbb{R} \to \text{Aut}(M, \tau)$ be a point-wise strongly continuous one-parameter group of automorphisms. Let δ be the infinitesimal generator of α, i.e. $\exp(t\delta) = \alpha_t$. For $f \in L^1(\mathbb{R})$, define the bounded operator $\alpha_f : M \to M$ by

$$\alpha_f(x) = \int_{-\infty}^{\infty} f(s)\alpha_s(x)\, ds.$$

It can be checked that if $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then

$$\delta \circ \alpha_f(x) = -\alpha_{f'}(x).$$

Also, if $x \in M \cap D(\delta)$, then we have

$$\alpha_t(x) - x = \int_0^t \delta\circ, \alpha_s(x)\, ds = \int_0^t \alpha_s(\delta(x))\, ds.$$

THEOREM 5.1. *Suppose that for every $\varepsilon > 0$, there exists $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$ and $\sup_{x\in(M)_1} \|\alpha_f(x) - x\|_2 \leq \varepsilon/4$. Then α_t converges $\|\cdot\|_2$-uniformly to the identity on $(M)_1$ as $t \to 0$.*

Proof. We need only show for every $\varepsilon > 0$ that there exists some $\eta > 0$ such that for all $t < \eta$, $\sup_{x\in(M)_1} \|\alpha_t(x) - x\|_2 \leq \varepsilon$. Let $\tilde{x} = \alpha_f(x)$. We have $\|\alpha_t(x) - x\|_2 \leq \|\alpha_t(\tilde{x}) - \tilde{x}\|_2 + \varepsilon/2$. Since $\delta \circ \alpha_f$ is defined everywhere, $\delta \circ \alpha_f : M \to L^2(M, \tau)$ is bounded. In fact, $\|\delta \circ \alpha_f\| \leq \|f'\|_{L^1}$. Now, since $\tilde{x} \in D(\delta)$, we have $\alpha_t(\tilde{x}) - \tilde{x} = \int_0^t \alpha_s(\delta(\tilde{x}))\, ds$. Hence, $\|\alpha_t(\tilde{x}) - \tilde{x}\|_2 \leq t\|f'\|_{L^1}$. Choosing $\eta = \varepsilon(2\|f'\|_{L^1})^{-1}$ does the job. □

COROLLARY 5.2. *If $\varphi_t = \exp(-t\delta^*\delta)$ converges uniformly to the identity as $t \to 0$, then so does α_t.*

Proof. Let $f_t(s) = (1/\sqrt{4\pi t}) \exp(-s^2/4t)$; then $\varphi_t(x) = \int_{-\infty}^{\infty} f_t(s)\alpha_s(x)\, ds$ follows by completing the square. □

5.2. *Tensor products of derivations.* We describe here the notion of a tensor product of derivations; see also [**29**, §6].

Consider N_i, $i \in I$, a family of finite von Neumann algebras with normal faithful traces τ_i. If $\delta_i : N_i \to \mathcal{H}_i$ is a family of closable real derivations into Hilbert bimodules $\mathcal{H}_i$ with domains $D(\delta_i)$, then we may consider the dense $*$-subalgebra $D(\delta) = \bigotimes_{i\in I}^{\text{alg}} D(\delta_i) \subset N = \overline{\bigotimes}_{i\in I} N_i$.

We denote by $\hat{N}_j$ the tensor product of the N_i's obtained by omitting the jth index, so that we have a natural identification $N = \hat{N}_j \mathbin{\overline{\otimes}} N_j$ for each $j \in I$. Let $\mathcal{H} = \bigoplus_{j\in I} \mathcal{H}_j \mathbin{\overline{\otimes}} L^2(\hat{N}_j)$, which is naturally a Hilbert bimodule because of the identification $N = \hat{N}_j \mathbin{\overline{\otimes}} N_j$.

The tensor product of the derivations δ_i, $i \in I$, is defined to be the derivation $\delta = \bigotimes_{i\in I} \delta_i : D(\delta) \to \mathcal{H}$ which satisfies

$$\delta\Bigl(\bigotimes_{i\in I} x_i\Bigr) = \bigoplus_{j\in I}\Bigl(\delta_j(x_j) \bigotimes_{i\in I, i\neq j} x_i\Bigr).$$

This is well defined, as only finitely many of the x_i's are not equal to 1 and hence the right-hand side is a finite sum.

If $\Phi_i^t = \exp(-t\delta_i^*\overline{\delta_i})$ is the semigroup deformation associated to δ_i, then one easily checks that the semigroup deformation associated to δ is $\Phi^t = \bigotimes_{i\in I} \Phi_i^t : N \to N$. A similar formula holds for the resolvent deformation. Note that by viewing the Hilbert bimodule associated to Φ^t and using the usual 'averaging trick' (e.g. [**32**, Theorem 4.2]), it follows that Φ^t will converge uniformly in $\|\cdot\|_2$ to the identity on $(N)_1$ if and only if each Φ_t^i converges uniformly in $\|\cdot\|_2$ to the identity on $(N_i)_1$ and moreover this convergence is uniform in $i \in I$.

5.3. *Derivations from generalized Bernoulli shifts.* We use here the notation in §2.1 above. Given a real Hilbert space $\mathcal{H}$, we consider the new Hilbert space $\mathcal{H}' = \mathbb{R}\Omega_0 \oplus \mathcal{H}$. If $\xi \in \mathcal{H}$ is a non-zero element, we denote by P_ξ the rank-one projection onto the subspace $\mathbb{R}\xi$. We denote by $\tilde{\mathcal{H}}$ the tensor product (complex) Hilbert space $\mathcal{H} \otimes \mathfrak{S}(\mathcal{H}')$.

Let $N \in \mathbb{N} \cup \{\infty\}$ be the dimension of $\mathcal{H}$ and consider an orthonormal basis $\beta = \{\xi_n\}_{i=1}^N$ for $\mathcal{H}$. We then define a left action of A, the von Neumann algebra generated by the spectral projections of $s(\xi)$, $\xi \in \mathcal{H}$, on $\tilde{\mathcal{H}}$ such that for each $\xi \in \mathcal{H}$, $s(\xi)$ acts on the left (as an unbounded operator) by

$$\ell_\beta(s(\xi)) = \mathrm{id} \otimes s(\xi).$$

We also define a right action of A on $\tilde{\mathcal{H}}$ such that for each $\xi \in \mathcal{H}$, $s(\xi)$ acts on the right by extending linearly the formula

$$r_\beta(s(\xi))(\xi_n \otimes \eta) = P_{\xi_n}(\xi) \otimes S(\Omega_0)\eta + \xi_n \otimes s(\xi - P_{\xi_n}(\xi))\eta \tag{1}$$

for each $1 \le n \le N$, $\eta \in \mathfrak{S}(\mathcal{H}')$.

These formulas define unbounded self-adjoint operators on $\tilde{\mathcal{H}}$ in general; however, by functional calculus they extend to give commuting normal actions of A on $\tilde{\mathcal{H}}$.

Moreover, if $T \in \mathcal{O}(\mathcal{H}) \subset \mathcal{O}(\mathcal{H}')$, then we have for any $\xi \in \mathcal{H}$

$$\ell_{T\beta}(s(T\xi)) = \ell_{T\beta}(\sigma_T(s(\xi))) = \mathrm{Ad}(T \otimes T^{\mathfrak{S}})\ell_\beta(s(\xi)).$$

Also,

$$r_{T\beta}(s(T\xi)) = r_{T\beta}(\sigma_T(s(\xi))) = \mathrm{Ad}(T \otimes T^{\mathfrak{S}})(r_\beta(s(\xi))).$$

From here on, we will denote the left action of A on $\tilde{\mathcal{H}}$ by $\ell_\beta(a)x = a \cdot_\beta x$ and the right action by $r_\beta(a)x = x \cdot_\beta a$. By extending the formulas above to A, we have the following lemma.

LEMMA 5.3. *Using the notation above, consider the inclusion $\mathcal{O}(\mathcal{H}) \subset \mathcal{U}(\tilde{\mathcal{H}})$ given by $T \mapsto \tilde{T} = T \otimes T^{\mathfrak{S}}$. Then, for each $T \in \mathcal{O}(\mathcal{H})$, $x, y \in A$, and $\tilde{\xi} \in \tilde{\mathcal{H}}$, we have $\tilde{T}(x \cdot_\beta \tilde{\xi} \cdot_\beta y) = \sigma_T(x) \cdot_{T\beta} (\tilde{T}\tilde{\xi}) \cdot_{T\beta} \sigma_T(y)$.*

Remark 5.4. While we will not use this in the following, an alternate way to view the A–A Hilbert bimodule structure on $\tilde{\mathcal{H}}$ is as follows. Given our basis $\beta = \{\xi_n\}_{n=1}^N \subset \mathcal{H}$, consider the probability space $(X, \mu) = \Pi_n(\mathbb{R}, g)$, where g is the Gaussian measure on $\mathbb{R}$. We can identify $A = L^\infty(X, \mu)$, and we denote by $\pi_n \in L^2(X, \mu)$ the projection onto the nth copy of $(\mathbb{R}, g)$, so that the π_n's are I.I.D. Gaussian random variables.

We embed $\mathcal{H}$ into $L^2(X, \mu)$ linearly by the map η such that $\eta(\xi_n) = \pi_n$. Given an orthogonal transformation $T \in \mathcal{O}(\mathcal{H})$, we associate to T the unique measure-preserving automorphism $\sigma_T \in \mathrm{Aut}(A)$ such that $\sigma_T(\eta(\xi)) = \eta(T\xi)$ for all $\xi \in \mathcal{H}$.

For each k, we denote

$$A_k = \left(\bigotimes_{n<k} L^\infty(\mathbb{R}, g)\right) \otimes (L^\infty(\mathbb{R}, g) \otimes L^\infty(\mathbb{R}, g)) \otimes \left(\bigotimes_{n>k} L^\infty(\mathbb{R}, g)\right),$$

and we view $L^2(A_k)$ as an A–A bimodule so that

$$(\otimes_n a_n) \cdot x = \left(\bigotimes_{n<k} a_n \otimes (a_k \otimes 1) \otimes \bigotimes_{n>k} a_n\right) x$$

and

$$x \cdot (\otimes_n a_n) = x\left(\bigotimes_{n<k} a_n \otimes (1 \otimes a_k) \otimes \bigotimes_{n>k} a_n\right)$$

for $x \in L^2(A_k)$.

Consider the A–A Hilbert bimodule $\bigoplus_k L^2(A_k)$, and note that it is canonically identified with the Hilbert space $\mathcal{H} \otimes L^2(A_1) \cong \mathcal{H} \otimes L^2(\mathbb{R}, g) \otimes L^2(A) \cong \tilde{\mathcal{H}}$ in a way which preserves the A–A bimodule structure. Under this identification, the inclusion $\mathcal{O}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H} \otimes L^2(\mathbb{R}, g) \otimes L^2(A))$ becomes $T \mapsto T \otimes \mathrm{id} \otimes \sigma_T$.

We now consider the algebra $A_0 \subset L^2(A)$ of square summable operators generated by $s(\xi)$, $\xi \in \mathcal{H}$, and define a derivation δ_β (compare with [**53**]) on A_0 by setting

$$\delta_\beta(s(\xi)) = \xi \otimes \Omega \in \tilde{\mathcal{H}}$$

for each $\xi \in \mathcal{H}$. Note that the formula for $\delta_\beta(s(\xi))$ does not depend on the basis β, but the bimodule structure that we are imposing on $\mathcal{H}$ does depend on β. If $\xi_0, \xi_1, \ldots, \xi_k \in \beta$ such that ξ_0 is orthogonal to the vectors $\xi_1, \ldots, \xi_k$, then it follows that $\delta_\beta(s(\xi_1) \cdots s(\xi_k))$ is an $s(\xi_0)$-central vector and hence by induction on k it follows that δ_β is well defined. Also, since δ_β extends to a bounded operator on $\overline{\mathrm{sp}}\{s(\xi_1) \cdots s(\xi_k) \mid \xi_1, \ldots, \xi_k \in \mathcal{H}\}$ for each k, it follows that δ_β is a closable operator and if we still denote by δ_β the closure of this operator we have that $x \mapsto \|\delta_\beta(x)\|^2$ is a quantum Dirichlet form on $L^2(A)$ (see [**12, 45, 46**]).

In particular, it follows from [**12**] that $D(\delta_\beta) \cap A$ is a weakly dense $*$-subalgebra and $\delta_{\beta|D(\delta_\beta)\cap A}$ is a derivation.

Note that if we identify $\tilde{\mathcal{H}}$ with $\bigoplus_k L^2(A_k)$ as above, then δ_β can also be viewed as the tensor product derivation $\delta_\beta = \bigotimes_k \delta_k$, where $\delta_k : L^2(\mathbb{R}, g) \to L^2(\mathbb{R}, g) \mathbin{\overline{\otimes}} L^2(\mathbb{R}, g)$ is the difference quotient derivation for each k, i.e. $\delta_k(f)(x, y) = (f(x) - f(y))/(x - y)$.

LEMMA 5.5. *Using the above notation, δ_β is a densely defined closed real derivation, $s(\mathcal{H}) \subset D(\delta_\beta)$, $\delta_\beta \circ s : \mathcal{H} \to \tilde{\mathcal{H}}$ is an isometry, and, for all $T \in \mathcal{O}(\mathcal{H})$, $\sigma_T(D(\delta_\beta)) = D(\delta_{T\beta})$ and $\delta_{T\beta}(\sigma_T(a)) = \tilde{T}(\delta_\beta(a))$ for all $a \in D(\delta_\beta)$.*

Proof. The facts that $s(\mathcal{H}) \subset D(\delta_\beta)$ and that $\delta_\beta \circ s$ is an isometry follow from the formula $\delta_\beta(s(\xi)) = \xi \otimes \Omega$ above.

Moreover, for $\xi \in \mathcal{H}$, we have

$$\delta_{T\beta}(\sigma_T(s(\xi))) = T\xi \otimes \Omega = (T \otimes T^{\mathfrak{S}})(\xi \otimes \Omega) = \tilde{T}\delta_\beta(s(\xi)).$$

By Lemma 5.3, this formula then extends to A_0 and, since $\tilde{T}$ acts on $\tilde{\mathcal{H}}$ unitarily and A_0 is a core for δ_β, we have $\sigma_T(D(\delta_\beta)) = D(\delta_{T\beta})$ and this formula remains valid for $a \in D(\delta_\beta)$. □

Given an action of a countable discrete group Γ on a countable set S, we may consider the generalized Bernoulli shift action of Γ on $(X, \mu) = \Pi_{s\in S}(\mathbb{R}, g)$ given by $\gamma(r_s)_{s\in S} = (r_{\gamma^{-1}s})_{s\in S}$. If we set $\mathcal{H} = \ell^2 S$ and consider the corresponding representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, then the generalized Bernoulli shift can be viewed as the Gaussian action corresponding to π. Moreover, we have that the canonical basis $\beta = \{\delta_s\}_{s\in S}$ is invariant to the representation, i.e. $\pi_\gamma \beta = \beta$ for all $\gamma \in \Gamma$.

In this case, by Lemma 5.5, we have that $D(\delta_\beta)$ is σ_γ invariant for all $\gamma \in \Gamma$ and $\delta_\beta(\sigma_\gamma(a)) = \tilde{\pi}_\gamma(\delta_\beta(a))$ for all $\gamma \in \Gamma$, $a \in D(\delta_\beta)$, where $\tilde{\pi} : \Gamma \to \mathcal{U}(\tilde{\mathcal{H}})$ is the unitary representation given by $\tilde{\pi} = \pi \otimes \pi^{\mathfrak{S}}$. If we denote by $N = A \rtimes \Gamma$ the corresponding group-measure space construction, then, using Lemma 5.3, we may define an N–N Hilbert bimodule structure on $\mathcal{K} = \tilde{\mathcal{H}} \otimes \ell^2\Gamma$ which satisfies

$$(au_{\gamma_1})(\xi \otimes \delta_{\gamma_0})(bu_{\gamma_2}) = (a \cdot_\beta (\tilde{\pi}_{\gamma_1}\xi) \cdot_\beta \sigma_{\gamma_1\gamma_0}(b)) \otimes \delta\gamma_1\gamma_0\gamma_2$$

for all $a, b \in A$, $\gamma_0, \gamma_1, \gamma_2 \in \Gamma$, and $\xi \in \tilde{\mathcal{H}}$. We may then extend δ_β to a closable derivation $\delta : *\text{-Alg}(D(\delta_\beta) \cap A, \Gamma) \to \mathcal{K}$ such that $\delta(au_\gamma) = \delta_\beta(a) \otimes u_\gamma$ for all $a \in D(\delta_\beta)$, $\gamma \in \Gamma$.

As above, we denote by $\zeta_\alpha : N \to N$ the unital, symmetric, completely positive resolvent maps given by $\zeta_\alpha = (\alpha/(\alpha + \delta^*\overline{\delta}))^{1/2}$ for $\alpha > 0$.

Note that if M is a finite von Neumann algebra, then we let Γ act on M trivially and we may extend the derivation δ to $(A \overline{\otimes} M) \rtimes \Gamma \cong (A \rtimes \Gamma) \overline{\otimes} M$ by considering the tensor product derivation of δ with the trivial derivation (identically 0) on M. In this case, the corresponding deformation of resolvent maps is just $\zeta_\alpha \otimes \text{id}$.

LEMMA 5.6. *Consider Ioana's deformation α_t on A corresponding to a generalized Bernoulli shift as described above in §3.2. If M is a finite von Neumann algebra and $B \subset (A \overline{\otimes} M) \rtimes \Gamma$ is a subalgebra such that ζ_α converges uniformly to the identity on $(B)_1$ as $\alpha \to 0$, then α_t converges uniformly to the identity on $(B)_1$ as $t \to 0$.*

Proof. The infinitesimal generator of Ioana's deformation cannot be identified with δ, as the α_t's will converge uniformly on the algebra generated by $s(\xi)$ for each $\xi \in \beta$, and ζ_α will not have this property. However, it is not hard to check using the fact that both derivations arise as tensor product derivations that if ζ_α^0 are the resolvent maps corresponding to the infinitesimal generator of α_t, then we have the inequality $\tau(\zeta_\alpha(a)a^*) \leq 2\tau(\zeta_\alpha^0(a)a^*)$ for all $a \in A$. Hence, the lemma follows from [**29**, Lemma 2.1] and Corollary 5.2 above. □

Remark 5.7. It can be shown in fact that the deformation coming from the derivation above, Ioana's deformation, and the s-malleable deformation from the Gaussian action are successively weaker deformations. That is to say, one deformation converging uniformly on a subset of the unit ball implies that the next deformation must also converge uniformly.

When we restrict the bimodule structure on $\mathcal{K}$ to the subalgebra $L\Gamma$, we see that this is exactly the bimodule structure coming from the representation $\tilde{\pi} = \pi \otimes \pi^{\mathfrak{S}}$; this gives rise to the following lemma.

LEMMA 5.8. *Using the notation above, given $H < \Gamma$ we have the following:*

(1) *${}_{LH}\mathcal{K}_{LH}$ embeds into a direct sum of coarse bimodules if and only if $\pi_{|H}$ embeds into a direct sum of left regular representations;*
(2) *${}_{LH}\mathcal{K}_{LH}$ weakly embeds into a direct sum of coarse bimodules if and only if $\pi_{|H}$ weakly embeds into a direct sum of left regular representations;*
(3) *${}_{LH}\mathcal{K}_{LH}$ has a stable spectral gap if and only if $\pi_{|H}$ has a stable spectral gap;*
(4) *${}_{LH}\mathcal{K}_{LH}$ is a mixing correspondence if and only if $\pi_{|H}$ is a mixing representation;*
(5) *${}_{LH}\mathcal{K}_{LH}$ is weakly mixing if and only if $\pi_{|H}$ is weakly mixing.*

6. L^2-rigidity and $\mathcal{U}_{\text{fin}}$-cocycle superrigidity

In this section, we use the tools developed above to prove $\mathcal{U}_{\text{fin}}$-cocycle superrigidity for the Bernoulli shift action, which we view as the Gaussian action corresponding to the left regular representation.

To prove that a cocycle untwists, we use the same general setup as Popa in [**38**]. In particular, we use the fact that for a weakly mixing action, in order to show that a cocycle untwists it is enough to show that the corresponding s-malleable deformation converges uniformly on the 'twisted' subalgebra of the crossed product algebra. The main difference in our approach is that to show that the s-malleable deformation converges uniformly it is enough by Lemma 5.6 to show that the deformation coming from the Bernoulli shift derivation converges uniformly. This allows us to use the techniques developed in [**26, 28–30**] to analyze the cocycle on the level of the base space itself rather than the exponential of the space, where the properties can be somewhat hidden.

THEOREM 6.1. *Let Γ be a countable discrete group. If $L\Gamma$ is L^2-rigid, then the Bernoulli shift action with diffuse core of Γ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

Proof. Let $\mathcal{G} \in \mathcal{U}_{\text{fin}}$; then $\mathcal{G} \subset \mathcal{U}(M)$ as a closed subgroup, where M is a finite separable von Neumann algebra. Let $c : \Gamma \times X \to \mathcal{G}$ be a cocycle, where X is the probability space of the Gaussian action. Consider $A = L^\infty(X)$, and $\omega : \Gamma \to \mathcal{U}(A \overline{\otimes} M)$ given by $\omega_\gamma(x) = c(\gamma, \gamma^{-1}x)$ the corresponding unitary cocycle for the action $\tilde{\sigma}_\gamma = \sigma_\gamma \otimes \text{id}$. Note that $\omega_{\gamma_1\gamma_2} = \omega_{\gamma_1}\tilde{\sigma}_{\gamma_1}(\omega_{\gamma_2})$ for all $\gamma_1, \gamma_2 \in \Gamma$. Here we view a unitary element in $A \overline{\otimes} M$ as a map from X to $\mathcal{U}(M)$ (see [**38**] for a detailed explanation).

As noted above, the Bernoulli shift action with diffuse core is precisely the Gaussian action corresponding to the left regular representation; hence, by Lemma 5.8, we have that as an $L\Gamma$–$L\Gamma$ Hilbert bimodule $\mathcal{K}$ embeds into a direct sum of coarse correspondences. If we denote by $\widetilde{L\Gamma}$ the von Neumann algebra generated by $\{\tilde{u}_\gamma\} = \{\omega_\gamma u_\gamma\}$, then the bimodule structure of $\widetilde{L\Gamma}$ ($\cong L\Gamma$) on $\mathcal{K}$ is the same as the bimodule structure of $L\Gamma$ on the correspondence coming from the representation $\gamma \mapsto \text{Ad}(\omega_\gamma) \circ \tilde{\pi}_\gamma$ on $\tilde{\mathcal{H}} \otimes L^2M$. The $A \overline{\otimes} M$ bimodule structure on $\tilde{\mathcal{H}} \otimes L^2M = \mathcal{H} \otimes \mathfrak{S}(\mathcal{H}') \otimes L^2M$ decomposes as a direct sum of bimodules $\mathcal{H} \otimes \mathfrak{S}(\mathcal{H}') \otimes L^2M = \bigoplus_{\xi \in \beta} \mathfrak{S}(\mathcal{H}') \otimes L^2M$, where the bimodule structure on each copy of $\mathfrak{S}(\mathcal{H}') \otimes L^2M$ is given by equation (1), and under this decomposition we have $\text{Ad}(\omega_\gamma) \circ \tilde{\pi}_\gamma = \pi_\gamma \otimes (\text{Ad}(\omega_\gamma) \circ \pi_\gamma^{\mathfrak{S}})$. Therefore, by Fell's absorption principle, this representation is an infinite direct sum of left regular representations; hence, we have that $\mathcal{K}$ also embeds into a direct sum of coarse correspondences when $\mathcal{K}$ is viewed as an $\widetilde{L\Gamma}$–$\widetilde{L\Gamma}$ Hilbert bimodule.

Since $L\Gamma$ is L^2-rigid, we have that the corresponding deformation ζ_α converges uniformly to the identity map on $(\widetilde{L\Gamma})_1$; by Lemma 5.6, we have that a corresponding s-malleable deformation also converges uniformly to the identity on $(\widetilde{L\Gamma})_1$. Thus, by [**38**, Theorem 3.2], the cocycle ω is cohomologous to a homomorphism. □

We end this paper with some examples of groups for which the hypothesis of Theorem 6.1 is satisfied.

It follows from [**29**] that if N is a non-amenable II_1 factor which is non-prime, has property Gamma, or is w-rigid, then N is L^2-rigid. We include here another class of

L^2-rigid finite von Neumann algebras; this class includes the group von Neumann algebras of all generalized wreath product groups $A_0 \wr_X \Gamma_0$, where A_0 is an infinite abelian group and Γ_0 does not have the Haagerup property, or Γ_0 is a non-amenable direct product of infinite groups. This is a special case of a more general result which can be found in [**30**].

THEOREM 6.2. *Let Γ be a countable discrete group which contains an infinite normal abelian subgroup and either does not have the Haagerup property or contains an infinite subgroup Γ_0 such that $L\Gamma_0$ is L^2-rigid; then $L\Gamma$ is L^2-rigid.*

Proof. We will use the same notation as in [**29**]. Suppose that (M, τ) is a finite von Neumann algebra with $L\Gamma \subset M$, and $\delta : M \to L^2M \overline{\otimes} L^2M$ is a densely defined closable real derivation.

Since the maps η_α converge point-wise to the identity, we may take an appropriate sequence α_n such that the map $\phi : \Gamma \to \mathbb{R}$ given by $\phi(\gamma) = \Sigma_n 1 - \tau(\eta_{\alpha_n}(u_\gamma)u_\gamma^*)$ is well defined. If the deformation η_α does not converge uniformly on any infinite subset of Γ, then the map ϕ is not bounded on any infinite subset and hence defines a proper, conditionally negative-definite function on Γ, showing that Γ has the Haagerup property.

Therefore, if Γ does not have the Haagerup property, then there must exist an infinite set $X \subset \Gamma$ on which the deformation η_α converges uniformly. Similarly, if $\Gamma_0 \subset \Gamma$ is an infinite subgroup such that $L\Gamma_0$ is L^2-rigid, then we have that the deformation η_α converges uniformly on the infinite set $X = \Gamma_0$.

Let $A \subset \Gamma$ be an infinite normal abelian subgroup. If there exists an $a \in A$ such that $a^X = \{xax^{-1} \mid x \in X\}$ is infinite, then we have that the deformation η_α converges uniformly on this set and, by applying the results in [**29**], it follows that η_α converges uniformly on $A \subset LA$. Since A is a subgroup in $\mathcal{U}(LA)$ which generates LA, it then follows that η_α converges uniformly on $(LA)_1$ and hence also on $(L\Gamma)_1$, since A is normal in Γ.

If $a \in A$ and a^X is finite, then there exists an infinite sequence $\gamma_n \in X^{-1}X$ such that $[\gamma_n, x] = e$ for each n. Thus, if a^X is finite for each $a \in A$, then, by taking a diagonal subsequence, we construct a new sequence $\gamma_n \in X^{-1}X$ such that $\lim_{n\to\infty}[\gamma_n, a] = e$. Since η_α also converges uniformly on $X^{-1}X$, we may again apply the results in [**29**] to conclude that η_α converges uniformly on A and hence on $(L\Gamma)_1$ as above. □

It has been pointed out to us by Ioana that in light of [**7**, Corollary 1.3] the above argument is sufficient to show that for a lattice Γ in a connected Lie group which does not have the Haagerup property, we must have that $L\Gamma$ is L^2-rigid.

We also show that L^2-rigidity is stable under orbit equivalence. The proof of this uses the diagonal embedding argument of Popa and Vaes [**43**].

THEOREM 6.3. *Let $\Gamma_i \curvearrowright (X_i, \mu_i)$ be free ergodic measure-preserving actions for $i = 1, 2$. If the two actions are orbit equivalent and $L\Gamma_1$ is L^2-rigid, then $L\Gamma_2$ is also L^2-rigid.*

Proof. Suppose that $L\Gamma_2 \subset M$ and $\delta : M \to \mathcal{H}$ is a closable real derivation such that $\mathcal{H}$ as an $L\Gamma_2$ bimodule embeds into a direct sum of coarse bimodules. Let $N = L^\infty(X_1, \mu_1) \rtimes L\Gamma_1 = L^\infty(X_2, \mu_2) \rtimes L\Gamma_2$ and consider the $N \overline{\otimes} M$ bimodule $\tilde{\mathcal{H}} = L^2N \overline{\otimes} \mathcal{H}$. If we embed N into $N \overline{\otimes} M$ by the linear map α which satisfies $\alpha(au_\gamma) = au_\gamma \otimes u_\gamma$ for all

$a \in L^\infty(X_2, \mu_2)$, and $\gamma \in \Gamma_2$, then when we consider the $\alpha(N)$–$\alpha(N)$ bimodule $\tilde{\mathcal{H}}$ we see that this bimodule is contained in a direct sum of the bimodule $L^2\langle \alpha(N), \alpha(L^\infty(X_1, \mu_1))\rangle$ coming from the basic construction of $(\alpha(L^\infty(X, \mu)) \subset \alpha(N))$. Indeed, this follows because the completely positive maps corresponding to left- and right-bounded vectors of the form $1 \otimes \xi \in L^2 N \,\overline{\otimes}\, \mathcal{H}$ are easily seen to live in $L^2\langle \alpha(N), \alpha(L^\infty(X_1, \mu_1))\rangle$.

The $\alpha(N)$–$\alpha(N)$ bimodule $L^2\langle \alpha(N), \alpha(L^\infty(X_1, \mu_1))\rangle$ is an orbit equivalence invariant and is canonically isomorphic to the bimodule coming from the left regular representation of Γ_1 (see for example [**34**, §1.1.4]). It therefore follows that $\tilde{\mathcal{H}}$ when viewed as an $\alpha(L\Gamma_1)$ bimodule embeds into a direct sum of coarse bimodules.

We consider the closable derivation $0 \otimes \delta : N \,\overline{\otimes}\, M \to \tilde{\mathcal{H}}$ as defined in §5.2 and use the fact that $L\Gamma_1$ is L^2-rigid to conclude that the corresponding deformation $\mathrm{id} \otimes \eta_\alpha$ converges uniformly on the unit ball of $\alpha(N)$ (note that $\mathrm{id} \otimes \eta_\alpha$ is the identity on $\alpha(L^\infty(X_1, \mu_1)) = \alpha(L^\infty(X_2, \mu_2))$). In particular, $\mathrm{id} \otimes \eta_\alpha$ converges uniformly on $\{\alpha(u_\gamma) \mid \gamma \in \Gamma_2\}$, which shows that η_α converges uniformly on $\{u_\gamma \mid \gamma \in \Gamma_2\}$. As this is a group which generates $L\Gamma_2$, we may then use a standard averaging argument to conclude that η_α converges uniformly on the unit ball of $L\Gamma_2$ (see for example [**32**, Theorem 4.1.7]). □

Remark 6.4. The above argument will also work to show that the 'L^2-Haagerup property' (see [**29**]) is preserved by orbit equivalence. In particular, this gives a new way to show that the von Neumann algebra of a group Γ which is orbit equivalent to free groups is solid in the sense of Ozawa [**24**], i.e. $B' \cap L\Gamma$ is amenable whenever $B \subset L\Gamma$ does not have minimal projections. Solidity of group von Neumann algebras for groups which are orbit equivalent to free groups was first shown by Sako [**44**].

We also note that by [**11**], any group which is orbit equivalent to a free group will have the complete metric approximation property. It will no doubt follow by using the techniques in [**26**] that the von Neumann algebra of a group Γ which is orbit equivalent to a free group will be strongly solid, i.e. $\mathcal{N}_{L\Gamma}(B)''$ is amenable whenever $B \subset L\Gamma$ does not have minimal projections.

Examples of groups which are orbit equivalent to a free group can be found in [**4, 17**].

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