

# Optimal Configurations on the Sphere and Other Manifolds

Abstracts

Vanderbilt University, Department of Mathematics  
Nashville, Tennessee, USA

May 17–20, 2010

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**Numerical Verification Method for Well Conditioned Spherical  $t$ -Designs**

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This talk presents a numerical verification method for well conditioned spherical designs with  $N \geq (t+1)^2$  points, as proposed in [1]. A set  $\mathcal{X}_N$  of  $N$  points on the unit sphere is a spherical  $t$ -design if the average of any polynomial of degree at most  $t$  over the sphere is equal to the average value of the polynomial over  $\mathcal{X}_N$ . A well conditioned spherical  $t$ -design is computed by maximizing the determinant of the matrix  $H_t(\mathcal{X}_N) := \mathbf{Y}_t(\mathcal{X}_N)\mathbf{Y}_t(\mathcal{X}_N)^T$  subject to the constraint that  $\mathcal{X}_N$  is a spherical  $t$ -design. Here  $\mathbf{Y}_t$  is a matrix of orthonormal spherical harmonics of degree  $\leq t$  at the point of  $\mathcal{X}_N$ . A spherical design is characterized as a set of solutions of a system of underdetermined nonlinear equations. Based on the computational existence proofs for spherical  $t$ -designs with  $N = (t+1)^2$  in [3], we give an algorithm for constructing well conditioned spherical  $t$ -designs with  $N \geq (t+1)^2$ . Moreover, using a preconditioning method, we give a sharp error bound for  $\log \det(H_t(\mathcal{X}_N))$ . The numerical verification method provides narrow intervals which contain well conditional spherical  $t$ -designs and the values of the determinant. Computations are conducted by using the numerical verification code [2] and the MATLAB toolbox INTLAB [4].

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[2] X. CHEN, A. FROMMER AND B. LANG, *Computational existence proof for spherical  $t$ -designs*, Preprint, The Department of Applied Mathematics, The Hong Kong Polytechnic University, August, 2009.

[3] X. CHEN AND R. S. WOMERSLEY, *Existence of solutions to systems of underdetermined equations and spherical designs*, SIAM J. Numer. Anal., 44 (2006), pp. 2326–2341.

[4] S. M. RUMP, *INTLAB – INTerval LABoratory*, in Developments in Reliable Computing, T. Csendes, ed., Dordrecht, 1999, Kluwer Academic Publishers, pp. 77–104.

## **On one extremal property of a regular simplex and its applications to adaptive mesh generation**

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Adaptive spline approximation (or interpolation) takes into account local variations in the behavior of the given function, adjusts the geometry of mesh elements (on which spline is constructed) depending on it, and hence yields the smaller error of approximation.

In this talk we show that the  $L_p$ -error of asymmetric linear approximation of the quadratic function  $Q(\mathbf{x}) = \sum_{j=1}^d x_j^2$  on simplices in  $\mathbb{R}^d$  of fixed volume is minimized on regular simplices. As an application we obtain a way to generate asymptotically optimal triangulations for asymmetric approximation of functions from  $C^2(D)$  with positive curvature.

## **Distance avoiding sets on the sphere and other manifolds**

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The computation of the maximal density of distance avoiding sets on the sphere and other manifolds is a difficult problem, related to classical questions in geometry such as the determination of the kissing number in dimension  $n$  or of the chromatic number of the Euclidean space. In this talk I will discuss a general framework leading to a computationally easier estimate for this number, which goes back to classical methods in graph theory and combines linear programming with cones and harmonic analysis of the isometry group of the manifold. I will also discuss the possibilities to tighten this estimate.

## **Euclidean designs and cubature formulas for spherical symmetric measures on $\mathbb{R}^n$**

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We first discuss the concepts of Euclidean  $t$ -designs and tight Euclidean  $t$ -designs. Then we survey the current status of the classification problem of tight Euclidean  $t$ -designs on  $p$  concentric spheres for small  $p$ . In particular, we classified tight Euclidean 5- and 7-designs on 2 concentric spheres. Recently, we have succeeded in proving the non-existence of tight Euclidean 9-designs on 2 concentric spheres in  $\mathbb{R}^n$  for all  $n \geq 3$ . We outline the proof of this result. Also, we point out that this implies the non-existence of so called minimal cubature formulas of degree 9 (in the sense of Cools-Schmid) for any spherically symmetric measure on  $\mathbb{R}^n$  with any  $n \geq 3$ .

(Remark: This abstract is for a talk in the special session: Cubature on the sphere)

## Rotationally Invariant Quadratures for the Sphere

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In the paper “Rotationally Invariant Quadratures for the Sphere”, Proc. R. Soc. A, v. 465, 2009, we have introduced a numerical method for constructing quadratures invariant under the icosahedral group, integrating exactly all  $(N + 1)^2$  linearly independent functions in a rotationally invariant subspace of maximal degree and order  $N$ . The nodes of these quadratures are nearly uniformly distributed and the number of nodes is only marginally more than the optimal  $(N + 1)^2/3$  nodes. Using these quadratures, we discretize the reproducing kernel on a rotationally invariant subspace to construct an analogue of Lagrange interpolation on the sphere. In this representation coefficients are the function values at the quadrature nodes and expansion involves a single function centered and mostly concentrated at nodes of the quadrature. Although it provides a much better localization than spherical harmonic expansions, we also develop a representation using functions that are localized even further.

New quadratures eliminate clustering and singularities due to the coordinate system and provide a path to efficient solution methods in many applications, for example, in acoustic and electromagnetic scattering problems posed as integral equations and involving integration over a sphere as well as in problems of quantum chemistry.

We note that quadratures invariant under other symmetry groups may be constructed following the same approach. In many ways our results are just the first step in a program to develop practical computational methods in a variety of applications. Our construction significantly extends availability of quadratures for the sphere with discrete group symmetry. So far we have constructed quadratures up to  $N = 210$  and are working on several extensions of the approach.

## On asymptotic bounds for spherical designs

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We will present our recent results on the upper bounds for the minimal cardinality of a spherical  $t$ -design on  $S^d$ , where  $d$  is fixed and  $t \rightarrow \infty$ . Both constructive and unconstructive approaches will be discussed.

## **Optimal recovery of certain classes of smooth multivariate functions**

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Let  $\mathcal{L}^*$  be the hexagonal lattice on the plane,  $\mathcal{L}$  be a two-dimensional sublattice of  $\mathcal{L}^*$ , and  $n = kN^2$ , where  $N$  is a positive integer and  $k$  is the cardinality of the group  $\mathcal{L}^*/\mathcal{L}$ . For the class of two-variate  $\mathcal{L}$ -periodic functions with uniformly bounded second derivative in any direction, we show that the best positions of  $n$  nodes for the optimal global recovery problem are at points of the lattice  $\frac{1}{N}\mathcal{L}^*$ . The optimal recovery problem is considered here in the worst case setting with the error being estimated in the uniform norm. For analogous classes of functions periodic with respect to  $d$ -dimensional lattices ( $d \geq 3$ ), we obtain asymptotic behavior (with an implicit constant) of the worst-case error of the optimal recovery algorithm and describe an asymptotically optimal sequence of node configurations. To study this problem we use known results on the most economical covering.

## **Two-dimensional condensed matter**

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In a wide variety of physical systems and over many length and energy scales one encounters ordered phases of matter on surfaces. The nature of the order has several novel features not encountered in the corresponding flat space systems, including the presence of various types of topological defects in the ground state that would normally only be excited states, and the instability of otherwise stable states. I will discuss the origin, structure and stability of these defect arrays as a function of the geometry and topology of the surface, the nature of the long range order itself and the number of ordering units.

## **Asymptotic Behavior of Minimal Discrete Riesz Energy on Curves and Spheres: Conjectures and Numerical Evidence**

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Given an  $N$ -point configuration in the Euclidean space  $\mathbb{R}^p$ , its Riesz  $s$ -energy is given by the sum of reciprocal  $s$ -powers of all mutual distances in this system. We discuss the asymptotical behavior (as  $N \rightarrow \infty$ ) of the minimal  $N$ -point Riesz  $s$ -energy of a compact set  $A$  in  $\mathbb{R}^p$ . The leading term is well understood in the potential-theoretical case (i.e. the set  $A$  has positive  $s$ -capacity) and known for a quite general class of sets  $A$  (e.g.  $d$ -rectifiable manifolds) in the singular case  $s \geq d$ . In this talk we present conjectures for the second term for curves and spheres and compare available numerical data. This is joint work with Ed Saff and Doug Hardin.

## **A Computational Examination of Stable Configurations on the Sphere**

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Many computational methods for finding a configuration of  $N$  points on  $\mathbf{S}^2$  that minimizes a pairwise energy rely entirely on differentiation. The presence of configurations that are stable but not minimal hinders these methods. Therefore an understanding of the nature and the number of these stable configurations is of value. We present a method for generating configurations that meet a criteria for stability with respect to the Riesz  $s$ -energy. Additionally we present graph-theoretic techniques for characterizing these configurations. The resulting data are used to estimate the growth of the number of stable configurations with  $N$  and to understand how the asymptotic expansion of energy with respect to  $N$  depends on the nature of the configurations.

LA-UR 10-01608

## The Forces Method in Fekete Problems

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The Fekete Problem consists in the minimization of energy functionals, depending on the relative distance between  $N$  points, under general constraints. The minimization of the electrostatic potential energy of  $N$  charges on a surface, or the constrained maximization of the product of the distances between  $N$  points are the best known examples.

In this work, we analyze several applications of the algorithm, which we call *the Forces Method*, to the Fekete Problem. Specifically, we describe the main properties of the algorithm to the search for Fekete Points in general manifolds. When we consider the logarithmic energy on the 2-sphere, we show that a local minima can be identified with an average cost of  $O(N^{2.8})$ . We present some experiments we performed at the supercomputing centers CESGA and BSC.

In addition, we apply the Forces Method to the resolution of the Principle of Least Action of a mechanical system. If a system occupies at two instants positions defined by two sets of values of the coordinates, the system moves between these positions in such a way that the integral of Action takes the least possible value.

If we discretize the time interval between these two instants, then we can consider the coordinates of the bodies at intermediate instants (virtual points) as the unknowns of the problem. Then we obtain an approximation of the stationary trajectories of the Action, which are the solutions to Newton's equations, by considering a Fekete Problem for the virtual points forming these trajectories. Specifically, we consider a set of virtual forces that acts on the virtual points, namely the kinetic component (the mass times the acceleration for each point at each instant) and the potential component (Newtonian forces acting on the virtual particle).

This methodology allows us to apply the Forces Method to the intermediate positions so that the effective component of the virtual forces system tends to 0. Therefore, we obtain an approximation of the trajectories that are solutions to the motion equations on any manifold, with initial and final fixed positions. Unlike forward Euler methods, this implicit method allows us to consider large time intervals and hence obtain, with a low cost, good approximations of the trajectories. Moreover, these solutions are stable under time refinement. We present here some properties of solutions to the Principle of Least Action that show their analogy with solutions to the Fekete Problem.

A further step consists in applying the Forces Method to problems in atomic scale. The consideration of molecules implies to use a potential energy that is more complex than the gravitatory one. In addition to a Coulombian potential, we must now consider a Lennard Jones potential together with a set of potentials that are due to the biochemical relations between atoms (basically, they are functionals of the bond lengths, of the bond angles and of the dihedral angles).

Because the repulsive forces are more difficult to treat, if we wish to avoid collisions, it is necessary to adjust the time of the trajectory to be proportional to a power, given by the dominant kernel, of the minimum distance between atoms. Applying the Forces Method for the protein folding problem, we obtain the path followed in the process of folding of the atoms, which leave from a starting position and arrive at the folded position.

### **Some experiments on packing and stacking of non-spherical objects**

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Non-spherical objects can frustrate or enable ordered structures. We present experiments on the random packing of tetrahedra which suggest they produce less translational and orientational order than spheres. Further we slightly generalize the concept of isostaticity from counting contacts or neighbors to counting constraints, e.g. three constraints for face to face contacts, appropriate for objects with faces and edges. We show that randomly packed tetrahedra are isostatic with 12 constraints, twice the 6 degrees of freedom per particle. Separately we present results on using geometrical lock and key interactions to organize particles with different complementary geometry.

### **An Asymptotic Analysis of the Mean First Passage Time for Narrow Escape Problems for the Sphere**

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The mean first passage time (MFPT) is calculated for a Brownian particle in a spherical domain in  $\mathbb{R}^3$  that contains  $N$  small non-overlapping absorbing windows, or traps, on its boundary. For the unit sphere, the method of matched asymptotic expansions is used to derive an explicit three-term asymptotic expansion for the MFPT for the case of  $N$  small locally circular absorbing windows. The third term in this expansion, not previously calculated, depends explicitly on the spatial configuration of the absorbing windows on the boundary of the sphere. The three-term asymptotic expansion for the average MFPT is shown to be in very close agreement with full numerical results. The average MFPT is shown to be minimized for trap configurations that minimize a certain discrete variational problem. This variational problem is closely related to the well-known optimization problem of determining the minimum energy configuration for  $N$  repelling point charges on the unit sphere. Numerical results, based on global optimization methods, are given for the optimum arrangements of the

centers of  $N$  circular traps on the boundary of the sphere. These optimum arrangements are compared with corresponding results for the classical Coulomb or logarithmic discrete energy functions.

### Ground states for the Gaussian core model in low dimensions

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The Gaussian core model is described by the Gaussian pair potential  $V(r) = \exp(-r^2)$ . Beyond its mathematical elegance, it is also used in physics to describe the effective interaction between the centers of mass of polymers. It is also a useful soft-core approximation (for low density) of the hard sphere model. The problem of determining the ground states of the Gaussian core model in dimensions beyond three is quite difficult.

I will describe joint work with Kumar and Schürmann which carried out computer simulations of Parrinello-Rahman dynamics for the Gaussian core model for various densities in dimensions two through eight, to experimentally determine the ground states. The results exhibit some surprising features, such as symmetry breaking and formal duality.

### On the energy of lattices and periodic sets in Euclidean spaces.

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Following Cohn and Kumar, we define the energy of a discrete set of points  $\Lambda$  in  $\mathbb{R}^d$  with respect to a potential  $f$  as

$$\lim_{R \rightarrow +\infty} 1/|\Lambda_R| \sum_{x,y \in \Lambda_R, x \neq y} f(|x-y|^2) \tag{1}$$

where  $\Lambda_R = \{x \in \Lambda, |x| \leq R\}$ .

When  $\Lambda$  is a lattice and  $f$  is an exponential (resp. an inverse power) this energy can be expressed *via* the theta series (resp. zeta function) of the lattice. In the above cited paper, Cohn and Kumar conjectured that the  $\mathbb{D}_4$ ,  $\mathbb{E}_8$ , and Leech lattice are *universally optimal* regarding energy, which means that they realize the minimal energy among all configurations with same point density in their dimension, for any potential  $f$  in the class of so-called *totally monotonic functions*.

In a joint work with Achill Schürmann, we obtained results giving some evidence to a local version of Cohn and Kumar’s conjecture. I will report on this work and also, as time allows, explain connections with problems from other fields (spherical designs, optimal metrics on Riemannian manifolds, numerical integration).

### **Random Close Packing of Disks and Spheres in Confined Geometries**

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We study the structure of many simulated random closing packings confined between two walls. Each packing consists of a binary mixture in equal number with a sizes ratio of 1.4. Our aim is to quantify how a confining boundary and the thickness between the boundaries alters the structure of randomly close packed disks in 2D and spheres in 3D. We find that confinement lowers the packing fraction, and induces heterogeneity in particle density where particles show strong layering near the wall. Both the particle density and the structure of the local packing show oscillations that decay outward from the wall. The decay in the oscillations is rapid, with a characteristic length scale less than the largest particle diameter. We have also developed a simple model for describing the decrease in packing fraction with confinement.

### **Separation results for optimal Riesz energy points on the sphere and axis-supported external fields**

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Let  $\omega_N := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\} \subset \mathbb{S}^d$  be an optimal configuration minimizing the Riesz energy  $\sum_{i \neq j} |\mathbf{x}_i - \mathbf{x}_j|^{-s}$ ,  $d - 2 \leq s < d$ . Considering the discrete and continuous minimum energy problems associated with the external field induced by a fixed point in  $\omega_N$ , the following separation result was proved by the speaker together with E. B. Saff

$$|\mathbf{x}_i - \mathbf{x}_j| \geq \frac{K_{s,d}}{N^{1/d}}, \quad K_{s,d} := \left( \frac{2\mathcal{B}(d/2, 1/2)}{\mathcal{B}(d/2, (d-s)/2)} \right)^{1/d},$$

where  $\mathcal{B}(x, y)$  denotes the Beta function. Motivated by this we extend the continuous minimal energy problem to axis-supported external fields. An interesting phenomenon occurs for  $s = d - 2$ . Finally, we discuss some generalizations of the method utilized in the separation result above.

## The Probabilistic Frame Potential and Random Choices of $n$ Points on the Sphere

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Finite unit norm tight frames (FNTFs) generalize orthonormal bases by allowing for redundancy. This more flexible concept has already proven useful in various applications. Choosing i.i.d. random points from the uniform probability distribution on the sphere is known to approximate a FNTF. We significantly extend the latter result by allowing for much more flexible choices of  $n$  points.

FNTFs were characterized as minimizers of the so-called frame potential. We introduce a new probabilistic frame potential and characterize its minimizers among all probability distributions on the sphere. Next, we verify that the collection of minimizers provides a versatile reservoir of probability distributions. We also explore the connections to a probabilistic version of spherical 2-designs. The probabilistic frame potential is then used to significantly weaken the requirements on the random choice of points to obtain an approximate FNTF: we allow for any distribution that minimizes the probabilistic frame potential and we remove the requirement that the points have to be identically distributed.

## Poisson summation and packing problems

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The Poisson summation formula on  $\mathbf{R}$  asserts that, for any sufficiently smooth and integrable function  $f$  on  $\mathbf{R}$ , the sum of  $f(x)$  over integers  $x$  equals the sum of  $\hat{f}(y)$  over integers  $y$ . Here  $\hat{f}$  is the Fourier transform of  $f$ , defined by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{2\pi ixy} dy.$$

This identity and its generalizations provide a remarkably versatile tool, used not just in real and complex analysis but also in number theory and the study of error-correcting codes. We concentrate on a generalization relevant to the study of packings in Euclidean spaces. Here  $f$  is a function on  $\mathbf{R}^n$ , the vector

$x$  ranges over a unimodular lattice (that is, the integer combinations of a basis of  $\mathbf{R}^n$  with determinant 1), and  $y$  ranges over the dual lattice (associated with the dual basis). We give several applications, both classical and modern, and some ideas and open questions that these uses of Poisson summation suggest.

### **Weighted thermodynamic formalism and applications**

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Let  $(X, T)$  and  $(Y, S)$  be two subshifts so that  $Y$  is a factor of  $X$ . For any asymptotically sub-additive potential  $\Phi$  on  $X$  and  $\mathbf{a} = (a, b) \in \mathbb{R}^2$  with  $a > 0$ ,  $b \geq 0$ , we introduce the notions of  $\mathbf{a}$ -weighted topological pressure and  $\mathbf{a}$ -weighted equilibrium state of  $\Phi$ . We setup the weighted variational principle and consider the uniqueness of weighted equilibrium states. Applications are given in non-conformal dynamical systems.

### **Geometry of optimal Riesz $s$ -energy configurations for large values of $s$**

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We investigate geometrical properties of Voronoi decompositions for optimal Riesz  $s$ -energy configurations of points for large values of  $s$  and show that the fraction of Voronoi cells that are ‘nearly’ regular hexagons is near 1 for large values of  $s$ .

## **A multiscale collocation method for solving partial differential equations on spheres.**

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Radial basis functions (RBFs) present a simple and effective way to construct approximate solutions to partial differential equations on spheres via collocation methods. The quality (in the supremum norm or in the  $L^2$  norm) of the approximation depends on the distribution of the centers of the RBFs used to define the approximate solution. Usually in practice the solution represents some physical quantities, which are available in many physical scales, and a solution using RBFs with a single scale may fail to capture these features. To overcome this, we propose a multiscale approximation scheme, in which the approximate solution is constructed after a multi-stage process, the residual of the current stage will be the target function for the next stage, and in each stage, an RBF with different scale will be used as basis functions. Even though the algorithm allows the collocation points to be scattered on the sphere, choosing the set of collocation points optimally will significantly improve the quality of the approximate solutions.

This is a joint work with I. Sloan (UNSW) and H. Wendland (Oxford).

## **Minimum energy configurations on the torus**

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In this talk I will discuss the ground state properties of axisymmetric toroidal crystals based on the elastic theory of topological defects on curved substrates. The ground state is analyzed as a function of the aspect ratio of the torus, which provides a non-local measure of the underlying Gaussian curvature, and the ratio of the defect core-energy to the Young modulus. Several structural features will be discussed, including an example of curvature-driven amorphization in the limit of the aspect ratio approaching one. The outcome of the elastic theory will be then compared with the results of a numerical study of a system of point-like particles constrained on the surface of a torus and interacting via a short range potential.

## Measurement of areas on a sphere using Fibonacci and latitude–longitude lattices

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The area of a spherical region can be measured by considering which sampling points of a spherical lattice are located inside or outside the region. This technique is frequently used for measuring the Earth coverage of satellite constellations, employing a latitude–longitude lattice.

This work (González, 2010) analyzes the numerical errors of these measurements, and shows that they could be greatly reduced if the Fibonacci lattice were used instead.

The spherical Fibonacci lattice is a mathematical idealization of natural spiral patterns with optimal packing. In the version used here (Dixon, 1987; Swinbank & Purser, 2006), the areas of the Voronoi cells are almost identical (Swinbank & Purser, 2006). This uniformity arises from three characteristics:

1. The angle turn between consecutive points along the generative spiral is the golden angle  $360^\circ\Phi^{-2} \simeq 137.5^\circ$ , or its complementary,  $360^\circ\Phi^{-1} \simeq 222.5^\circ$ , where  $\Phi = (1 + \sqrt{5})/2 \simeq 1.618$  is the golden ratio. Since this ratio is the “most irrational” number, periodicities or near-periodicities in the spiral arrangement are avoided, and clumping of the lattice points never occurs.
2. The generative spiral of the lattice is analogous to a Fermat spiral, which embraces an equal area per equal angle turn. Thus, each point is placed at a different latitude, in the center of a latitudinal band of equal area.
3. The first and last points are offset from the poles, leading to a more homogenous arrangement on the polar regions.

Using a Monte Carlo technique, the area of randomly located spherical caps is measured here using Fibonacci lattices and latitude–longitude lattices (with equally and unequally weighted point contributions, respectively).

The maximum root mean squared error is seen to decrease proportionally to  $P^{-3/4}$ , where  $P$  is the number of lattice points. This scaling is partially explained using arguments from similar problems on the plane. This error is about 40% larger in the latitude–longitude lattice, which is shown to arise from its poorer sampling efficiency.

The maximum absolute errors observed in the Monte Carlo simulation decrease more rapidly with  $P$  in the Fibonacci lattice than in its latitude–longitude counterpart. If, as is commonly the case, about a million lattice points are used for measuring Earth coverage, the maximum error would be an order of magnitude smaller with the Fibonacci lattice.

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## Discrete and continuous singular minimal energy on self-similar fractals

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Coauthors: Douglas Hardin and Edward Saff

Let  $f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $i = 1, \dots, k$  be a finite number of contracting similitudes. Then there is a unique compact set  $K$  such that

$$K = \bigcup_{i=1}^k f_i(K).$$

Conditions on the separation of the images  $f_i(K)$  like the open set condition allow the explicit computation of the Hausdorff-dimension  $d$  of  $K$ . Such conditions will also be needed in the context of energy.

We study different notions of energy on  $K$ . The main goal is to characterise the normalised Hausdorff-measure on  $K$  as the minimiser of such energy expressions.

1. **Continuous singular energy:** For a probability measure  $\mu$  on  $K$  we define

$$E(\mu) = \lim_{s \rightarrow d^-} (d - s) \int \int_{K \times K} \|x - y\|^{-s} d\mu(x) d\mu(y).$$

We can show that the Hausdorff-measure is the unique minimiser of this energy. We also study other notions of modified Riesz-energies for  $s = d$ .

2. **Discrete singular energy:** For a finite point set  $X \subset K$  and  $s \geq d$  define

$$E(X) = \sum_{x, y \in X, x \neq y} \|x - y\|^{-s}.$$

We study the limiting distribution of

$$\frac{1}{\#X} \sum_{x \in X} \delta_x$$

for point sets  $X$ , which minimise  $E(X)$  and  $\#X \rightarrow \infty$ .

## On the computation of spherical $t$ -designs by a new optimization approach based on fast spherical Fourier transforms

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In this talk we consider the problem of finding  $t$ -designs on the sphere  $\mathbb{S}^2$ . The concept of spherical  $t$ -designs was introduced by Delsarte, Goethals and Seidel [2] in 1977. There a spherical  $t$ -design on  $\mathbb{S}^2$  is defined as a finite set  $X_M = \{\mathbf{x}_1, \dots, \mathbf{x}_M\} \subset \mathbb{S}^2$  satisfying

$$\int_{\mathbb{S}^2} p(\mathbf{x}) d\mu_{\mathbb{S}^2}(\mathbf{x}) = \frac{4\pi}{M} \sum_{i=1}^M p(\mathbf{x}_i), \quad \text{for all } p \in \mathbb{P}_t,$$

where  $d\mu_{\mathbb{S}^2}$  is the surface measure on  $\mathbb{S}^2$  and  $\mathbb{P}_t$  is the space of all polynomials in  $\mathbb{R}^3$  with degree at most  $t$ . Here we exploit the equivalent characterization given by Sloan and Womersley in [4],

$$A_t(X_M) := \frac{1}{M^2} \sum_{n=1}^t \sum_{k=-n}^n \left| \sum_{i=1}^M Y_n^k(\mathbf{x}_i) \right|^2 = 0, \quad (2)$$

where  $Y_n^k$  are the spherical harmonics of degree  $n$  and order  $k$ . Since  $A_t \geq 0$ , the problem of finding a  $t$ -design  $X_M$  reduces to finding a minimum of  $A_t$ . We demonstrate how a nonlinear CG method on manifolds, cf. [5], is used to attack the problem (2). The proposed iterative algorithm benefits from the nonequispaced fast spherical Fourier transform [3]. We perform every iteration step in  $\mathcal{O}(t^2 \log^2 t + M)$  arithmetic operations with an overall amount of  $\mathcal{O}(t^2 + M)$  memory. We are satisfied numerically if the obtained minimum of  $A_t$  is at most a given accuracy  $\varepsilon > 0$ . Starting with some uniform distributions obtained in [1] we show for some choices  $t$  and  $M$  the performance of the algorithm and present numerical results even in the challenging case  $M \approx 1/2t^2$ .

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### **Potential theory and the asymptotics of ground state configurations.**

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I will discuss recent and classical results concerning the asymptotic properties (as  $N$  gets large) of ‘ground state’ configurations of  $N$  particles restricted to a compact set  $A$  of Hausdorff dimension  $d$  and interacting through through an inverse power law  $1/r^s$  for some  $s > 0$ .

If  $s < d$ , it is a classical result from Potential Theory that ground state configurations have limit distribution (as  $N$  approaches infinity) given by the (Reisz- $s$ ) equilibrium measure  $\mu_s$  on  $A$ , while the first order asymptotic growth of the energy of these configurations is given by the ‘transfinite diameter of  $A$ . In joint work with M. Calef we investigate the behavior of  $\mu_s$  as  $s$  approaches the critical value  $d$  (for  $s \geq d$ , there is no equilibrium measure). In the case that  $A$  is a fractal, the notion of ‘order two density introduced by Bedford and Fisher naturally arises.

As  $s$  becomes large, ground state configurations approach best-packing configurations on  $A$ . In joint work with S. Borodachov and E. Saff we show that for  $d$ -rectifiable sets  $A$  and  $s > d$  such configurations are asymptotically uniformly distributed on  $A$ .

### **Random Packings Under Stress**

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The properties of sphere packings have fascinated mathematicians and scientists for ages. Within the context of jamming there has been much recent interest in the mechanical stability of random sphere packings. I will discuss some ideas and first experimental results in which the concept of a jamming transition and the associated dramatic change in mechanical properties is exploited to design the stress response of amorphous granular materials. To illustrate this I will use examples from our work on randomly packed granular polymers (aka flexibly connected chains of spheres), shear thickening in suspensions, and jamming-based soft robots.

## Dense packings of regular tetrahedra

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The determination of the densest packings of regular tetrahedra (one of the five Platonic solids) is attracting great attention and a number of fascinating packing structures have emerged. In this talk, we will provide dense tetrahedron packings obtained from both theoretical considerations and numerical simulations, including the Welsh packing, icosahedral packing, wagon-wheel packings and dimer-uniform packings. The dimer-uniform packings contains the densest known tetrahedron packings with density  $\phi = \frac{4000}{4671} = 0.856347\dots$ . We will also provide arguments concerning the optimality of these packings.

## The *divide and concur* approach to packing

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We report on a novel, constraint-based heuristic search method for discovering dense packings of spherical and non-spherical particles. The *divide and concur* scheme uses a deterministic map in a non-physical configuration space as its search dynamic. Among the important features of the application of this scheme to packing problems are its aggressive variation of unit-cell parameters and the mutability of particle shapes. The generality and efficiency of the method, as demonstrated in several applications, and its highly non-physical dynamics make it a strong alternative to more conventional physics-inspired methods when optimal configurations are more of an interest than physical pathways. We present the main features of the *divide and concur* search method and some previous and recent results, including surprising results in the problems of tetrahedron and pentatope packing.

## Monotonicity results for N-body ground state energies

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The pair-specific ground state energy  $\varepsilon_g(N) := \mathcal{E}_g(N)/(N(N-1))$  of classical  $N$ -body systems is shown to grow monotonically in  $N$ . This furnishes a whole family of simple new tests for minimality of putative ground state energies  $\mathcal{E}_g^x(N)$  obtained through computer experiments. Inspection of several publicly available lists of such computer-experimentally obtained putative ground state energies  $\mathcal{E}_g^x(N)$  yielded several dozen instances of  $\mathcal{E}_g^x(N)$  which failed one of these tests; i.e., for those  $N$  one concludes that  $\mathcal{E}_g^x(N) > \mathcal{E}_g(N)$  strictly. Although the correct  $\mathcal{E}_g(N)$  is not revealed by this method, it does yield a better upper bound on  $\mathcal{E}_g(N)$  than  $\mathcal{E}_g^x(N)$  whenever  $\mathcal{E}_g^x(N)$  fails a monotonicity test. The surveyed  $N$ -body systems include in particular  $N$  point charges with 2- or 3-dimensional Coulomb pair interactions, placed either on the unit 2-sphere or on a 2-torus (a.k.a. Thomson, Fekete, or Riesz problems). We also present some related results for quantum  $N$ -body ground state energies.

## Energy minimization and connections with sphere packing and spherical codes.

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I will give a survey of recent progress in problems of energy minimization for discrete point sets in spheres, Euclidean spaces and some other manifolds. I will describe some connections to the spherical code and sphere packing problems, and some conjectures such as universal optimality of the  $A_2$ ,  $E_8$  and Leech lattices, and describe some results on compact spaces, as well as some applications of linear programming bounds to the inverse problem of potential energy minimization.

## Structure of digit sets of self-affine tiles

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The class of self-affine tiles  $T(A, \mathcal{D})$  in  $\mathbb{R}^n$  are generated by  $n \times n$  integer expanding matrix  $A$  together with certain digit sets  $\mathcal{D} \subset \mathbb{Z}^n$ . It has rich algebraic, geometric and topological properties. In this talk we will consider the structure of  $\mathcal{D}$  for  $T(A, \mathcal{D})$  to be a tile. Our concentration is mainly on the one-dimensional case, it involves with factorization of the cyclic groups and the cyclotomic polynomials. The higher dimensional case is mostly unknown.

## Greedy energy points in the presence of external fields

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In this talk we will describe some asymptotic properties of greedy energy sequences (or generalized Leja sequences) constructed under the influence of an external field. The algorithms that generate these sequences are based on an iterative procedure. One example is the following. Let  $m \geq 2$  be a fixed integer,  $A \subset \mathbb{R}^p$  an infinite closed set, and  $f : \mathbb{R}^p \rightarrow (-\infty, \infty]$  an external field. Then we construct inductively a sequence  $(a_i)_{i=1}^\infty$  by selecting the first  $m$  points  $a_1, \dots, a_m$  so that the expression

$$\sum_{1 \leq i < j \leq m} \frac{1}{|x_i - x_j|^s} + (m-1) \sum_{i=1}^m f(x_i)$$

attains its minimum on  $A^m$  for  $x_i = a_i$ ,  $1 \leq i \leq m$ , and for every integer  $N \geq 1$ , the points  $a_{mN+1}, \dots, a_{m(N+1)}$  are chosen to minimize the expression

$$\sum_{i=1}^m \sum_{l=1}^{mN} \frac{1}{|x_i - a_l|^s} + \sum_{1 \leq i < j \leq m} \frac{1}{|x_i - x_j|^s} + ((N+1)m-1) \sum_{i=1}^m f(x_i)$$

on  $A^m$ . Here  $s > 0$  is the Riesz parameter. Other related topics will be discussed, such as the computation of the equilibrium measure for radially symmetric functions in the Newtonian case.

## **Marcinkiewicz-Zygmund inequalities and Fekete arrays on the sphere**

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Fekete points are points that maximize a Vandermonde-type determinant that appears in the polynomial Lagrange interpolation formula. They are well suited points for interpolation formulas and numerical integration. We prove the asymptotic equidistribution of Fekete points on the sphere. This is done by showing their connection with other arrays of points, the Marcinkiewicz-Zygmund arrays and the interpolating arrays. Following the work of H. Landau and A. Beurling, we prove necessary geometric conditions for such arrays, in terms of certain densities defined on the sphere. These conditions imply the result for Fekete arrays.

This is a joint work with Joaquim Ortega-Cerdà from the Universitat de Barcelona.

## **Thermodynamics of sphere packings under different geometries**

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We used micron sized attractive colloidal spheres as building blocks to study sphere packings under different geometries. We first confined particles inside micro-wells and studied the structures and free energies of self-assembled 3-dimensional clusters. We found that highly symmetric clusters are strongly suppressed by rotational entropy, and the most stable clusters have anharmonic vibrational modes or extra bonds. Secondly, we encapsulated the particles inside emulsion droplets, and particles formed 2-dimensional close-packed monolayers on the curved interior walls of spherical droplets. We studied the topological properties of the packed structures and the related physics of two-dimensional nucleation/melting on spherical surfaces. We will discuss the connections between mathematical theory of sphere packing and thermodynamics of self-assembly for both experiments.

## Marcinkiewicz–Zygmund inequalities and quadrature on manifolds

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Let  $\mathbb{X}$  be a compact, connected, Riemannian manifold (without boundary),  $\rho$  be the geodesic distance on  $\mathbb{X}$ ,  $\mu$  be a probability measure on  $\mathbb{X}$ , and  $\{\phi_k\}$  be an orthonormal system of continuous functions,  $\phi_0(x) = 1$  for all  $x \in \mathbb{X}$ ,  $\{\ell_k\}_{k=0}^\infty$  be a nonincreasing sequence of real numbers with  $\ell_0 = 1$ ,  $\ell_k \uparrow \infty$  as  $k \rightarrow \infty$ ,  $\Pi_L$  be the span of  $\{\phi_j : \ell_j \leq L\}$ ,  $L \geq 0$ . We describe conditions to ensure an equivalence between the  $L^p$  norms of elements of  $\Pi_L$  with their suitably discretized versions. We also give intrinsic criteria to determine if any system of weights and nodes allows such inequalities. The results are stated in a very general form, applicable for example, when the discretization of the integrals is based on weighted averages of the elements of  $\Pi_L$  on geodesic balls rather than point evaluations.

## Fractal differential equations defined by iterated function systems with overlaps

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Coauthors: John F.-C. Chan, Jie Chen, Jiaxin Hu, and Ka-Sing Lau

We study spectral asymptotics of a class of Laplacians defined by iterated function systems with overlaps. We also study the computation of the eigenfunctions and eigenvalues of the Laplacians. We also study the wave equation. Part of this work is joint with J. Chan, J. Chen, J. Hu, and K.-S. Lau.

## A generalization of Larman-Rogers-Seidels theorem

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A finite subset  $X$  in the Euclidean space  $\mathbb{R}^d$  is called an  $s$ -distance set, if the size of the set of Euclidean distances of two distinct elements in  $X$  is equal to  $s$ . One of basic problems for  $s$ -distance sets is to determine the maximum cardinality of  $s$ -distance sets for fixed  $s$  and  $d$ . In 1977, Larman, Rogers, and Seidel proved that if the size of a 2-distance set is greater than  $2d + 2$ , then the ratio of two squared distances is  $k : (k - 1)$  where  $k$  is an integer. This result is a good constraint to determine the maximum cardinality of 2-distance sets.

In this talk, for any  $s$ , we give a generalization of this result. Moreover, we introduce some applications of this generalized theorem.

### **The Role of Packing in Determining Material Properties: From Hard Spheres to Ellipses and from Collapsed Polymers to Folded Proteins**

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In this talk, I will survey my recent computational and theoretical studies of static packings with different particle shapes and interactions, dimensionalities, boundary conditions, and constraints. In particular, I will highlight three interesting results: 1. The bulk shear rigidity of packings composed of ellipsoidal particles is much weaker than that for spherical particles; 2. Systems composed of ellipsoidal particles have intriguing nonequilibrium phases such as those with liquid-like rotational degrees of freedom (DOF), but frozen translational DOF, and logarithmic temporal relaxation of structural correlations; and 3. Chain connectivity constraints strongly affect the probabilities with which collapsed polymer packings occur.

### **Simple Universal Bounds for Chebyshev-type quadratures**

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A Chebyshev-type quadrature for a probability measure  $\sigma$  is a distribution which is uniform on  $n$  points and has the same first  $k$  moments as  $\sigma$ . We consider probability measures on an interval and give upper and lower bounds for the minimal number of points  $n$  required to achieve a given degree of accuracy  $k$ . Our bounds use only simple properties of  $\sigma$  and are applicable in wide generality. In particular, we obtain that any probability measure with bounded density has a Chebyshev-type quadrature with  $n$  at most exponential in  $k$ , and there exist such probability measures for which this bound is sharp, up to the constants involved. We apply our bounds to the construction of point sets on the sphere forming local approximate Chebyshev-type quadratures. Several open questions will be presented.

## Quadrature Rules on Spherical Triangles

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In this talk we present the construction of quadrature rules on arbitrary triangulations of the sphere which are exact for polynomials of some fixed degree. In a first step we study quadrature on some preassigned nodes so that we are able to compute integrals over triangles for arbitrary polynomials. In a second step we apply Cholesky decomposition methods to obtain the weights for scattered data. For our numerical tests we used Mathematica where we carried out all calculations in high accuracy or even with exact numbers. So we were able to overcome a lot of instability problems particularly for very small and thin triangles. Finally, we compare our local quadrature rules on triangulations and some small polynomial degree of exactness with global formulas on the whole sphere and high degree of polynomial exactness. Particularly, for clustered data the local methods seem to be better. Moreover we present some computations with some very local terrain data.

## Carleson measures and Logvinenko-Sereda sets on compact manifolds

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Given a compact Riemannian manifold  $M$ , we consider the space of functions of  $L^2(M)$ , generated by eigenfunctions of eigenvalue less than  $L \geq 1$ , associated to the Laplace-Beltrami operator on  $M$ . We study for which sequence of measures,  $\{\mu_L\}_L$  one has

$$C_1 \int_M |f|^2 dV \leq \int_M |f|^2 d\mu_L \leq C_2 \int_M |f|^2 dV$$

with constants independent of  $L$  and  $f$ . The inequality

$$\int_M |f|^2 d\mu_L \leq C_2 \int_M |f|^2 dV$$

defines the Carleson measures and we will present a geometric characterization of them. The converse inequality will be studied only for the special case  $d\mu_L = \chi_{A_L} dV$ , where  $\{A_L\}_L$  is a sequence of sets in  $M$ . In such case, when the inequality

$$\int_M |f|^2 dV \leq C_1 \int_{A_L} |f|^2 dV$$

is verified, we say that this sequence of sets is Logvinenko-Sereda. We will give a characterization of these sets.

### **Distribution of point charges with small discrete energy**

Igor Pritsker  
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We study the asymptotic equidistribution of points near arbitrary compact sets of positive capacity in  $\mathbb{R}^d$ ,  $d \geq 2$ . Our main tools are the energy estimates for Riesz potentials. We also consider the quantitative aspects of this equidistribution in the classical Newtonian and logarithmic cases. In particular, we quantify the weak\* convergence of discrete measures to the equilibrium measure, and give the estimates of convergence rates for discrete potentials to the equilibrium potential. These results are applied to the problems of Schur on the arithmetic means of zeros of polynomials with integer coefficients. We prove a generalization of the Erdős-Turán discrepancy theorem. Further applications include the estimates of growth for Fekete polynomials, convergence rates for discrete energy approximations to Robin's constant, and the distribution of Fekete points.

### **The Stability of Optimizers of Energy or Density**

Charles Radin  
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I will discuss the stability of energy and density optimizers. Such stability arises naturally in modelling the behavior of a variety of materials, such as crystals, quasicrystals, colloids, granular and crumpled matter. Physical experiments and computer simulations of models suggest provocative phenomena, which pose challenging mathematical problems.

## Observation of a shape-dependent random packing density maximum for colloidal ellipsoids

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Spherocylinders and other particle shapes such as ellipsoids have a maximal random packing density that, according to experiments [1] as well as simulations [2], uniquely depends on the particle aspect ratio. An intriguing prediction from simulations [2,3,4] is that, starting from the Bernal random sphere packing, the packing density first rises to a maximum for nearly spherical particles; and only beyond this maximum the random packing density monotonically decreases with aspect ratio due to the growth of the orientationally averaged excluded volume. We have investigated fluids of colloidal ellipsoids, quenched via ultra-centrifugation into random packings, to observe for the first time this density maximum for a colloidal system [5], which is located 12% above the Bernal sphere packing density.

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## Inductive construction of projective cubature formulas

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The cubature formulas on the projective spaces  $\mathbb{K}\mathbf{P}^{m-1}$ , where  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , are considered. Denote by  $N_{\mathbb{K}}(m, 2t)$  the minimal number  $n$  s.t. a projective cubature formula of index  $2t$  with  $n$  nodes on  $\mathbb{K}\mathbf{P}^{m-1}$  exists.

We give an inductive on  $m$  construction of such cubature formula. In some cases this allows us to improve the upper bounds for  $N_{\mathbb{K}}(m, 2t)$  known before.

This approach also yields some new cubature formulas over  $\mathbb{K}$  using some known ones over  $\mathbb{R}$ .

### **Sequences of point sets and scaled RBFs for multiscale approximation on the sphere,**

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Because physical phenomena on the earth's surface occur on many different length scales, it makes sense when seeking an efficient approximation to start with a crude approximation followed by a sequence of corrections on finer and finer scales. In the present talk, describing recent joint work with Q. Thong Le Gia and Holger Wendland, we make use of a sequence of point sets with decreasing mesh norm, and a sequence of (spherical) radial basis functions with finer and finer scales centered at the points, to obtain an approximation of any desired accuracy in an efficient and stable manner. While the idea of a multiscale scheme like this has appeared previously, for example in papers of Schaback, Narcowich/Schaback/Ward, Floater/Iske, and Hales/Levesley, and in books and papers of Freeden and colleagues, there seems to be no existing analysis of a multiscale approximation based on scaled versions of a single compactly supported RBF and scattered data, for either spherical or Euclidean regions. In this talk I shall outline the ideas behind our analysis of the multiscale scheme, illustrate the method with a problem from geophysics, and discuss how best to choose the sequences of point sets and scales.

### **From Packing Planes in Four-Space to Quantum Error-Correcting Codes**

Neil J. A. Sloane

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I will describe the route that took us from a new packing problem (looking for "codes" in Grassmann manifolds) to the solution of a new coding problem (codes for quantum error-correction). This began as a project with Ron Hardin and John Conway, but many others (Peter Shor, Rob Calderbank, Eric Rains, Gabriele Nebe, ...) have since been involved. There are also applications to medicine, to visualizing multi-dimensional data, and to wireless communications.

### **The strong thirteen spheres problem**

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The thirteen spheres problem is asking if 13 equal size nonoverlapping spheres in three dimensions can touch another sphere of the same size. This problem was the subject of the famous discussion between Isaac Newton and David Gregory in 1694. The problem was solved by Schutte and van der Waerden only in 1953. A natural extension of this problem is the strong thirteen spheres problem (or the Tammes problem for 13 points) which asks to find an arrangement and the maximum radius of 13 equal size nonoverlapping spheres touching the unit sphere. In the paper we give a solution of this long-standing open problem in geometry. Our computer-assisted proof is based on an enumeration of the so-called irreducible graphs.

### **Shift radix systems and tilings**

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We will talk about a class of dynamical systems, so-called shift radix systems. Shift radix systems form a generalization of many well-known notions of number systems like beta-numeration and canonical number systems. We will mainly focus on geometric properties of shift-radix systems. In particular, we will show that they admit tilings of the real vector space and study some properties of these tilings.

### **From Unusual Ground States to Packing Problems**

Salvatore Torquato

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Ground-state problems naturally arise in many fields, including physics, biology, materials science, and mathematics. A classical ground-state configuration of a system of interacting particles in  $d$ -dimensional Euclidean space is one that minimizes the system potential energy. In the laboratory, for example, such states can be produced by slowly cooling a liquid to a temperature of absolute zero, and usually the ground states are close-packed crystal structures. I will describe isotropic interactions with unusual crystal ground states in three dimensions (e.g., diamond crystal) as well as disordered ground states, a counterintuitive phenomenon. I then discuss the problem of determining the densest

packings of particles in  $d$  dimensions, which are closely related to ground-state problems. We provide the putative exponential improvement on a 100-year-old lower bound on the maximal packing density of spheres in high dimensions due to Minkowski. This suggests that disordered (rather than ordered) sphere packings may be the densest for sufficiently large dimension, implying the existence of disordered ground states for some continuous potentials. We also conjecture that the densest packings of the Platonic and Archimedean solids with central symmetry are given by their corresponding densest lattice packings. This is the analogue of Kepler's sphere-packing conjecture for these solids.

**Spherical  $t$ -designs with roughly  $t^2/2$  points and well-conditioned spherical  $t$ -designs.**

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Spherical  $t$ -designs are equal weight numerical integration rules for the sphere which are exact for all spherical polynomials of degree up to  $t$ . They are known to exist if the number of points  $N$  is large enough and that the lower bound on  $N$  of roughly  $t^2/4$  is not achievable for  $t \geq 3$ . It still not known that spherical  $t$ -designs with  $N = O(t^2)$  exists for all degrees  $t$ .

This talk looks at several variational characterisations based on functions with strictly positive Legendre coefficients and related systems of nonlinear equations. This provides computed spherical  $t$ -designs for  $t$  up to 138 and symmetric  $t$  designs for  $t$  up to 181. Moreover these computed spherical designs have good geometric properties.

When  $N$  is larger than the threshold around  $t^2/2$ , for example  $N = (t+1)^2$ , the extra degrees of freedom can be used to also optimise other criteria such as the determinant or condition number of the basis matrix. This is used to produce well-conditioned spherical designs.

## **Geophysical Modeling on the Sphere with Radial Basis Functions**

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Approximation on the sphere is fundamental to many problems in the geosciences. Classical approaches to these problems are based on expansions of spherical harmonics and/or tensor product methods on latitude/longitude based grids. The former are quite algorithmically complex, while the latter suffer from the notorious pole problem. Additionally, neither of the methods can be easily generalized to other manifolds. Radial Basis functions (RBFs), on the other hand, are algorithmically simple, suffer from no pole, and generalize to arbitrary geometries. Since RBFs do not depend on any grid and require no meshing, they can be naturally used in concert with optimal node configurations. We discuss three recent and non-trivial geophysical applications of RBFs on spherical domains with optimal node sets. The first is on the approximation and decomposition of tangent vector fields on the sphere (e.g. horizontal winds in the atmosphere). The second is the simulation of unsteady nonlinear flows on the sphere described by the shallow water equations. The third and final application is the simulation of thermal convection in a 3-D spherical shell, a situation of interest in modeling the earth's mantle.

## **Lower bound for the number of nodes of cubature formulas**

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The purpose of the talk is to discuss lower bounds for the number of nodes  $N$  of a cubature formula of degree  $2n - 1$  over a regular domain, including sphere, ball and cube. As an example, based on some recent results, we conjecture that  $N \geq n^3/4 + O(n^2)$  for a 3-dimensional cube, and perhaps also for 3-dimensional ball, instead of  $n^3/6 + O(n^2)$ .

## Large deviations for configurations of zeros on Riemann surfaces

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The  $N$  zeros of a polynomial of degree  $N$  in one complex variable define a configuration of  $N$  points on the Riemann sphere. If we endow the polynomials with an inner product and a Gaussian probability measure, then we obtain a random configuration of  $N$  points. It is known that the zeros of a random polynomial are close to an associated equilibrium measure for large  $N$  (Shiffman-Z, Bloom). In recent work with O. Zeitouni, we prove a large deviations principle for the empirical measure of zeros: the probability of deviation from equilibrium measure is exponentially small as measured by a certain rate function. In continuing work, the result is generalized to higher genus Riemann surfaces.

## Equilibrium problems for infinite dimensional vector potentials with external fields

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The lecture deals with a minimal energy problem in the presence of an external field  $\mathbf{f} = (f_i)_{i \in I}$  over noncompact classes of infinite dimensional vector measures  $\mu = (\mu^i)_{i \in I}$  in a locally compact space. The components  $\mu^i$  are positive measures (charges) normalized by  $\int g_i d\mu^i = a_i$  (where  $a_i$  and  $g_i$  are given) and supported by given closed sets  $A_i$  with the sign  $+1$  or  $-1$  prescribed such that  $A_i \cap A_j = \emptyset$  whenever  $\text{sign } A_i \neq \text{sign } A_j$ , and the law of interaction of  $\mu^i$ ,  $i \in I$ , is determined by the interaction matrix  $(\text{sign } A_i \text{sign } A_j)_{i,j \in I}$ . For all positive definite kernels satisfying Fuglede's condition of consistency between the vague (=weak\*) and strong topologies, sufficient conditions for the existence of equilibrium measures are established and properties of their uniqueness, vague compactness, and continuity under exhaustion of  $A_i$  by compact  $K_i$  are studied. Sharpness of the statement on the existence of equilibrium measures is discussed by providing examples of non-solvability. We also obtain variational inequalities for the  $\mathbf{f}$ -weighted equilibrium potentials, single out their characteristic properties, and analyze continuity of the equilibrium constants. Such results are new even for classical kernels in  $\mathbb{R}^n$ , which is important in applications. They can be found in:

N. Zorii, *Equilibrium problems for infinite dimensional vector potentials with external fields*, arXiv:0911.0901 (2009), 25 p.