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On subgroup distortion in finitely presented groups

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Abstract. It is proved that every computable function $G \to \mathbb{N} = \{0, 1, \dots\}$ on a group G (with certain necessary restrictions) can be realized up to equivalence as a length function of elements by embedding G in an appropriate finitely presented group. As an example, the length of g^n, the nth power of an element g of a finitely presented group, can grow as n^θ for each computable $\theta \in (0, 1]$. This answers a question of Gromov [2]. The main tool is a refined version of the Higman embedding established in this paper, which preserves the lengths of elements.

Bibliography: 10 titles.

§ 1. Introduction

Let G be a group with finite generating set $\mathcal{A} = \{a_1, \dots, a_m\}$. Then the *length* $|g| = |g|_{\mathcal{A}}$ of an element g is the *length* $\|W\|$ of the shortest word W on the alphabet $\mathcal{A}^{\pm 1}$ representing the element g. If G is isomorphically embedded in another group H with generating set $\mathcal{B}=\{b_1, \dots, b_n\}$, then we have the following inequality, which is easy to verify:

$$|g|_{\mathcal{B}} \leqslant c|g|_{\mathcal{A}} \tag{1.1}$$

with some constant $c = \max\{|a_1|_{\mathcal{B}}, \dots, |a_m|_{\mathcal{B}}\}$ independent of the element g. In particular, for two generating sets $\mathcal{A}$ and $\mathcal{B}$ of the group G there exist positive constants c_1 and c_2 such that

$$c_1|g|_{\mathcal{A}} \leqslant |g|_{\mathcal{B}} \leqslant c_2|g|_{\mathcal{A}} \tag{1.2}$$

for each $g \in G$.

The answer to the following question is given in [1]: what are the functions $l\colon G \to \mathbb{N} = \{0, 1, 2, \dots\}$ (where G is a fixed countable group G) that can be realized (up to equivalence) as the length functions $g \mapsto |g|_{\mathcal{B}}$ corresponding to an embedding of G in some group H with finite generating set $\mathcal{B}$. The appropriate concept of equivalence is in this case connected with inequalities (1.2). To be precise, two functions $l_1, l_2 : G \to \mathbb{N}$ are said to be *equivalent* if there exist positive constants c_1 and c_2 such that

$$c_1 l_1(g) \leqslant l_2(g) \leqslant c_2 l_1(g)$$

for arbitrary $g \in G$.

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In the next result of the present author the 'sufficiency' part is non-trivial.

Theorem 1 [1]**.** *Let $l\colon G \to \mathbb{N}$ be a function defined by an embedding of a group G in some group R with finite generating set $\mathcal{B} = \{b_1, \dots, b_n\}$, that is, $l(g) = |g|_{\mathcal{B}}$. Then*

- (D1) *$l(g) = l(g^{-1})$ for each $g \in G$, and $l(g) = 0$ if and only if $g = 1$;*
- (D2) *$l(gh) \leqslant l(g) + l(h)$ for $g, h \in G$;*
- (D3) *there exists a positive number a such that $\operatorname{card}\{g \in G \mid l(g) \leqslant r\} \leqslant a^r$ for each $r \in \mathbb{N}$.*

Conversely, for each group G and each function $l\colon G \to \mathbb{N}$ satisfying conditions (D1)–(D3) (*the* D*-condition for short*) *there exists an isomorphic embedding of G in some 2-generated group R with generating set $\mathcal{B} = \{b_1, b_2\}$ such that the function $g \mapsto |g|_{\mathcal{B}}$ is equivalent to l.*

In the particular case of the infinite cyclic group $G = \langle g \rangle$ one obtains, for example, groups $R_\alpha \geqslant G$ where the length of g^i grows as i^α for arbitrary preassigned $\alpha \in (0, 1]$. This answers the corresponding question of Gromov [2].

However, one can encounter various effects of cyclic subgroup distortion already for finitely presented groups and even for groups with a single relation . For example, the function $|c^i|$ in the group $\langle a, b, c \mid [a, b] = c,\ [a, c] = [b, c] = 1 \rangle$ grows as $\sqrt{i}$. In other words, the central cyclic subgroup $\langle c \rangle$ has a quadratic distortion. (As a measure of distortion of a subgroup G of a group H one takes the function $\operatorname{disto}(r) = \max_{|g|_{\mathcal{B}} \leqslant r} |g|_{\mathcal{A}}$, although this function does not enable us to reconstruct the equivalence class of the function $g \mapsto |g|_{\mathcal{B}}$ on G.) In the group $\langle a, b \mid aba^{-1} = b^2 \rangle$ the function $|b^i|$ grows as $\log i$, that is, the cyclic subgroup $\langle b \rangle$ has exponential distortion. An example of a distortion stronger than multi-exponential along with other examples can be found in [3] and [2]. A series of examples of embeddings of specially constructed non-cyclic groups in finitely presented groups was constructed in [2] with functions $\operatorname{disto}(r) \sim r^q$ for arbitrary rational exponents $q \geqslant 1$.

For that reason, Gromov [2] emphasizes the following question in the 'realization problem': find all possible distortions of cyclic subgroups in finitely presented groups. It is obviously impossible to 'realize' an arbitrary function satisfying the D-condition under such a restriction in the way it was done in Theorem 1. One reason is the continuum cardinality of the set of all equivalence classes of functions satisfying the D-condition. (All the functions i^α are pairwise inequivalent). However, we succeed in the realization of all 'reasonably defined' distortion functions and, moreover, not just for cyclic subgroups.

Theorem 2. *Let l be a computable function with properties* (D1)–(D3) *on a group G. Then G can be isomorphically embedded into some finitely presented group H in such a way that the function l is equivalent to the restriction of $|\ \ |_H$ to G.*

Here l is said to be computable if for each representation of an arbitrary element g as a word on some generating set one can algorithmically compute $l(g)$. It is not necessary to assume for this result that G has finitely many generators. Further, we can use the notation $|\ \ |_H$ regardless of the particular choice of a finite generating set in H if we consider functions up to equivalence. Another obvious observation is that the lengths of elements in a group H are computable with respect to any finite generating set if the word problem is algorithmically soluble in H.

Just as conjectured in [1], the key for the proof is the following refinement of Higman's theorem which makes it possible to derive Theorem 2 from Theorem 1.

Theorem 3. *Let R be a group with a finite generating set $\mathcal{A}$ and a recursively enumerable set of (defining) relations. Then there exists an isomorphic embedding of R in some group H with a finite generating set $\mathcal{B}$ and a finite system of defining relations such that $|g|_{\mathcal{A}} = |g|_{\mathcal{B}}$ for each $g \in R$.*

Note that Theorem 3 has been recently proved for semigroups by Birget [5].

It is not difficult to improve slightly the statement of Theorem 3 by a proper choice of a universal group H. Bearing in mind Lemma 2.2 one can take the same group (independent of G) in Theorem 2 as a receptacle of all possible 'computable distortions' of finitely presented subgroups.

Theorem 4. *There exists a group U with a finite generating set $\mathcal{B}$ and a finite set of defining relations that has the following property. For an arbitrary group H with a finite generating set $\mathcal{A}$ and a recursively enumerable set of defining relations there exists an isomorphic embedding $H \hookrightarrow U$ such that $|g|_{\mathcal{A}} \leqslant |g|_{\mathcal{B}}$ for all $g \in H$.*

Our plan in this paper is as follows. In §2 we recall a construction from [1] and explain why the group R in Theorem 1 is recursively presented in the case of a computable function l. In §3 we discuss several useful properties of hyperbolic planar graphs, that is, planar graphs with vertices of sufficiently high degree.

Relations of the group H in Theorem 3 are presented in §4. They are modifications of those suggested in Rotman's book [6], Chapter 12. In this book Britton's proof of the Novikov – Boon theorem (with later improvements due to Boon, Collins and Miller taken into account) is followed by Aanderaa's proof [7] of Higman's embedding theorem. (The author uses this opportunity to thank E. Rips and S. V. Ivanov for drawing his attention to Aanderaa's proof.)

The difference is a lengthier relation (4.10) as compared with [6]. This enables us to substantiate and use certain hyperbolic properties of van Kampen diagrams over a presentation in §4. In the construction of groups with prescribed Dehn function, Birget, Rips, and Sapir were first to demonstrate the advantage of such a modification. However, in contrast to [8], we keep the Turing machine of [6]; in particular, it remains deterministic.

The geometric explanations given in [6], Chapter 12 and, more systematically, in [8], explore the following fact. Relations typical for HNN and similar extensions equip van Kampen diagrams with a certain structure by stratifying them into various 'strips'. We present the corresponding terminology in §5. In §6 we prove a lemma on the impossibility of multiple intersections for certain types of strips in 'hub'-free diagrams, that is, in diagrams without 2-cells corresponding to relations (4.10).

In §7 we give necessary geometrical results on minimal diagrams over the modified Novikov–Boon group; for instance, we discuss why two hubs can have at most one common 'spoke'.

Relations of Aanderaa type appear in §8. Although the group under consideration is finitely presented, it turns out that to get rid of strips 'strongly enveloping hubs' in minimal diagrams one should admit infinitely many relations to the list of defining relations, while the minimality of a diagram should be understood to be with regard to the grading, that is, the ranks of cells.

In §9 we must also temporarily admit infinitely many relations to the list of defining relations. This is necessary in order to eliminate pairs of hubs connected by a large number of spokes in minimal diagrams.

To prove Theorem 3 we construct in §10 four subgroups intermediate between R and H and carry out several intermediate comparisons of the lengths of elements.

We use all the main lemmas in §11. In particular, we use the important disjointness property of 'derived ovals' in hyperbolic graphs, which helps us to find estimates for the lengths of boundary pieces in a minimal diagram.

Our main results are proved in §12.

The techniques from §§ 8-12 are also applicable in the situation of [8] when defining relations are built on the basis of other machines and without the 'quadratic' letter 'x'. A joint paper using these techniques and the results of [8] is now in preparation. We prove there that the word problem in a group G is soluble by a non-deterministic machine in polynomial time if and only if the group G can be embedded in a group with a polynomial isoperimetric function. The properties of the machines constructed in [8] give us the following refinement of Theorem 3: the word problem in the group H is soluble if and only if it is soluble in R.

§ 2. Embedding in a recursively presented group

To deduce Theorem 2 from Theorems 1 and 3 one must explain why the group R has a recursively enumerable set of defining relations in Theorem 1 if the function l with the D-condition on the group G is computable. To this end we now repeat briefly the construction of the embedding in Theorem 1 of [1].

First, we construct algorithmically an auxiliary set of positive words $\mathcal{M}$ on 2 variables with certain necessary properties of the small cancellation condition kind. This set is *exponential*, that is, there exists a number $c > 1$ such that for each $i \geqslant i_0$ there exist at least c^i words of length $\leqslant i$ in $\mathcal{M}$, where the quantities c and i_0 can be explicitly found.

Further, in Lemma 5 we construct an injective map

$$g \mapsto X_g \in \mathcal{M}, \tag{2.1}$$

where $g \in G\backslash 1$. This map defines an embedding of G in a group H on two generators defined by all relations of the type $X_{g_1} X_{g_2} \cdots X_{g_n} = 1$ with $g_1 g_2 \ldots g_n = 1$ in G. We now repeat the proof of Lemma 5 in [1] while fixing our attention on the effective search for the words X_g.

Lemma 2.1. *Let* $\mathcal{M}$ *be an exponential set of words on a finite alphabet* $\{a_1, \ldots, a_m\}$. *Then for a computable function* $l\colon G \to \mathbb{N}$ *with the* D*-condition one can find a constant* $d = d(\mathcal{M}, l)$ *and an effectively defined map* (2.1) *such that*

$$l(g) \leqslant \|X_g\| < dl(g), \qquad g \in G \setminus 1. \tag{2.2}$$

Proof. By condition (D3) there exists a constant a such that

$$\operatorname{card} G_i < a^i \tag{2.3}$$

for each subset $G_i = \{g \in G \setminus 1 | l(g) \leqslant i\}$. Let c be the exponential constant of the set $\mathcal{M}$. Then one can choose a number $d > i_0$ such that

$$c^{di-1} - (2m+1)^i \geqslant a + a^2 + \dots + a^i, \qquad i = 1, 2, \dots. \tag{2.4}$$

Since the number of all words of length $\leqslant i-1$ is less than $(2m+1)^i$, the inequality (2.4) means that the set $\mathcal{M}$ has more than $a + \dots + a^i$ words with lengths in the segment $[i, di-1]$. Hence, one can (effectively) distinguish in $\mathcal{M}$ pairwise disjoint subsets $\mathcal{X}_{i1}, \dots, \mathcal{X}_{ii}$ consisting of $a, \dots, a^i$ words, respectively. (Here we can assume that a is an integer.) We perform this in such a way that all the words from the previously chosen subsets $\mathcal{X}_{i-1,1}, \dots, \mathcal{X}_{i-1,i-1}$ with lengths in the segment $[i, di-1]$ are included in $\mathcal{X}_{i1}, \dots, \mathcal{X}_{i,i-1}$, respectively. (Note that $G_0 = \varnothing$ by the D-condition.) Hence the subsets $\mathcal{X}_{ii}$, $i = 1, 2, \dots$ are pairwise disjoint.

Note that the word problem is algorithmically soluble in the group G by condition (D1) and the computability of l. Hence non-trivial elements in G (represented by words on some alphabet) can be effectively enumerated. Considering an arbitrary element $g = g_k$, we calculate the value $l(g) = i$ and associate it with X_g, the first word in the subset $\mathcal{X}_{ii}$ that has not been used before. There always exists such a word by condition (D3) and the equality $\operatorname{card} \mathcal{X}_{ii} = a^i$. This completes the proof of the lemma.

Lemma 2.2. *If the function l with the* D*-condition from Theorem* 1 *is computable, then the group R can be presented by a recursively enumerable set of defining relations.*

Proof.. As mentioned earlier, the word problem is algorithmically soluble in the group G. Hence, checking successively all equalities of the form $g_1 \cdots g_n = 1$ for various non-trivial factors from G and using Lemma 2.1 and the definition of R, we obtain an enumeration of the defining relations of R.

§ 3. Hyperbolic planar graphs

We consider a finite planar 2-complex, that is, a graph Γ in a plane with vertices $v_0, v_1, \dots, v_n$, where $n \geqslant 1$. The distinguished vertex v_0 is said to be *exterior*, and the others are *interior* vertices. We assume that the graph Γ is arranged in the plane in such a way that it has no 1-gons with vertices v_i and no 2-gons with vertices v_i and v_j for $i, j \geqslant 1$. There are also no loops with vertex v_0 (but it is quite possible that the vertex v_0 is connected with another vertex by several edges). We call such a graph an *l-graph*, $l \geqslant 6$, if the degree (of incidence) of each interior vertex is at least l.

The graph Γ can be put in the plane so that the vertex v_0 is indeed exterior, that is, can be connected with infinity by a curve disjoint from the edges of the graph Γ. Let p be a simple closed path formed by edges of Γ. Then one can speak about the domain O bounded by this path (or interior with respect to p). Each vertex o of p has some inner degree $d(o, p)$ relative to p, the number of edges from o inside O. We now present a well-known 'curvature formula' (Corollary 3.3 for $(6,3)$-maps in Chapter 5 of [9]).

Lemma 3.1. *Let* $o_1, \dots, o_s$ *be all consecutive vertices of a simple closed path* p *in an* l*-graph that does not pass through the exterior vertex. Then*

$$\sum_{i=1}^{s} \bigl(2 - d(o_i, p)\bigr) \geqslant 6 \tag{3.1}$$

(*that is, the average inner degree of the vertices* $o_1, \dots, o_s$ *is less than* 2).

The following property is also well known.

Lemma 3.2. *For each* l*-graph* Γ *there exists an interior vertex* o *of degree* $d \geqslant l$ *such that there exist at least* $d - 3$ *consecutive edges* (*in clockwise order*) *joining this vertex with the exterior vertex* v_0 *and there are no other vertices of* Γ *between adjacent edges.*

Proof. Let o_1 be an interior vertex in Γ. If the first assertion of the lemma fails, then o_1 can be joined with some interior vertex o_2 by an edge e_1. If the first assertion fails for o_2 as well, then we draw an edge e_2 from o_2 to some interior vertex o_3, and so on, until we create a simple closed path p.

Hence there exists a reduced closed path q without self-intersections (up to arbitrarily small deformations) that runs through interior vertices and bounds some maximal possible domain O. Because of maximality, q has a simple closed subpath q' that has at most one common vertex v with its complement q'' in q. However, it is clear from (3.1) that q' has at least one more vertex v' of inner degree at most 1. If the first assertion fails for this vertex again, then we can find an edge f_1 from v' to some interior vertex such that f_1 lies neither in O nor on q. Repeating this and finding edges $f_2, \dots$ one can define a path q_0 without self-intersections (consisting of these new edges and some edges of q) that bounds a larger domain than O.

Hence there exists a vertex o of the graph Γ satisfying the first assertion of the lemma. If there exist other vertices between some adjacent edges e' and e'' joining o and v_0, then the problem can be reduced to the subgraph lying between e' and e'' (containing v_0, but not o), which has fewer vertices. This completes the proof.

Lemma 3.3. *An* l*-graph is connected for each* $l = 6, 7, \dots$ *.*

Proof. Otherwise some connected component of the graph does not contain the exterior vertex. This contradicts the above lemma since $d - 3 \geqslant l - 3 \geqslant 3 > 0$.

Lemma 3.4. *Let* $d_0, d_1, \dots, d_n$ *be the degrees of the vertices* $v_0, v_1, \dots, v_n$ *in an* l*-graph* Γ*. Then* $d_0 \geqslant 6 - 6n + \sum_{i=1}^{n} d_i$*.*

Proof. This result holds for $n = 1$ since all the edges from v_1 must terminate at v_0. For $n > 1$ we may assume that the assertion of Lemma 3.2 holds just for the vertex $o = v_n$. We consider now a graph Γ' with vertices $v_0, \dots, v_{n-1}$ in which the edges of Γ joining $v_1, \dots, v_{n-1}$ with v_n are replaced by edges joining $v_1, \dots, v_{n-1}$ and v_0. (This can be done since v_n is connected with v_0 in Γ by Lemma 3.2.) Then Γ' is an l-graph and by Lemma 3.2 the degrees d_0 and d_0' of v_0 in Γ and in Γ' satisfy the inequality $d_0' - 3 \leqslant d_0 - (d_n - 3)$. Hence we obtain by the inductive hypothesis that

$$d_0 \geqslant d_0' + d_n - 6 \geqslant \sum_{i=1}^{n-1} d_i - 6(n-1) + 6 + d_n - 6 = \sum_{i=1}^{n} d_i - 6n + 6.$$

A curve without self-intersections in the standard hyperbolic plane that has constant (but not very large) curvature may be unbounded in both directions. Moreover, two such curves drawn orthogonally through the end-points of two segments with common origin do not intersect if these segments are sufficiently long and the angle between them is not too small. In the rest of this section we consider some discrete analogues of this hyperbolic phenomenon for finite graphs. They provide a basis for the estimates in §10.

Let e and e' be edges emanating from some interior vertex v of an l-graph Γ. Then all edges from v can be written in cyclic order as $e, f_1, \dots, f_{n_1}, e', f_1', \dots, f_{n_2}'$. We say that the edges e and e' form a *large angle* if $n_1 \leqslant 2n_2$ and $n_2 \leqslant 2n_1$.

A path $e_1 e_2 \dots e_\alpha$ in Γ is *weakly curved* if the end-points $(e_1)_+, \dots, (e_{\alpha-1})_+$ of its edges are interior vertices and the edges $(e_i)^{-1}$ and e_{i+1} form a large angle at the vertex $(e_i)_+ = (e_{i+1})_-$ for all $i = 1, \dots, \alpha - 1$.

Lemma 3.5. *If a weakly curved path p in an l-graph, $l \geqslant 6$, starts at an interior vertex $(e_1)_-$, then it cannot be closed.*

Proof. Reasoning by contradiction and choosing a path p of smallest length α we can assume that p is a simple path. In view of its weak curvature all vertices of the cycle p, except maybe one vertex, $p_- = p_+$, have inner degree at least 2 relative to p. However, this contradicts inequality (3.1).

Lemma 3.5 shows that a maximal possible extension (in both directions) of a weakly curved path in an l-graph has both end-points at the exterior vertex v_0.

Let x be a Jordan arc in the plane that has precisely two common points, x_- and x_+, with a graph Γ, and let both these points be vertices of the graph. We say that such an arc is Γ-*tame*. We consider a simple path p in Γ passing through some vertex v that is either x_- or x_+. All edges going out of v can be cyclically written out in the form $f_1, \dots, f_d$ in such a way that the arc x approaches v between f_d and f_1. This arc is said to be k-*separated* from p if the edges $f_1, \dots, f_k$ and $f_{d-k+1}, \dots, f_d$ (or their inverses) do not occur in p.

Two maximal weakly curved paths p and p' in an l-graph Γ are said to be *divergent* at vertices v and v' of it if for a pair of distinct interior vertices v and v' there exists a Γ-tame arc x joining these vertices but 2-separated from p and p'.

Lemma 3.6. *Two divergent paths in an l-graph Γ have no common interior vertices.*

Proof. In the above notations, let $\Gamma(x)$ be the graph coinciding with the graph Γ if the arc x forms a 2-gon with some edge e in Γ, or obtained from Γ by adding x as a new edge otherwise. It can be readily verified that $\Gamma(x)$ is also an l-graph.

Assume that p and p' have a common interior vertex. Then we can construct from pieces of p and p' and the path x (or the edge e) a simple closed path q in $\Gamma(x)$

that does not pass through the exterior vertex. By the weak curvature of p and p', there exist at most three vertices of q that can make a positive contribution to the left-hand side of inequality (3.1). These can be $x_- = v$, $x_+ = v'$, and the common vertex of p and p' . However, the inner degree of both x_- and x_+ relative to q is at least 1 (and even if we consider e in place of x in $\Gamma(x)$ this holds for e_+ and e_- because of the 2-separation condition in the definition of divergent paths). Hence the left-hand side of (3.1) is less than 6, in contradiction with Lemma 3.1. This proves the lemma.

According to Lemma 3.5 a maximal weakly curved path p in an l-graph Γ is simple and therefore partitions the plane into two domains O_1 and O_2 (inner and outer). If the number of edges from each vertex $v \neq v_0$ of p towards the domain O_1 (resp., towards the domain O_2) is at least eight more than towards O_2 (resp., O_1), then we say that the path p is an *oval* and set $O(p) = O_1$ (resp., $O(p) = O_2$). (It is easy to see that ovals can exist only for $l = 26$ or $l \geqslant 28$.)

A simple plane Jordan curve is said to be Γ-*transversal* if it does not pass through the vertices of a graph Γ and intersects transversally the edges of this graph. We say that a Γ-transversal curve *envelops* some interior vertex of the graph Γ if it intersects successively the edges $f_1, \dots, f_s$ issuing from v, where $s - 2$ is at least one-half of the degree of v, and moreover, if there exist no other vertices of the graph G in the 'sectors' formed by this curve and adjacent edges.

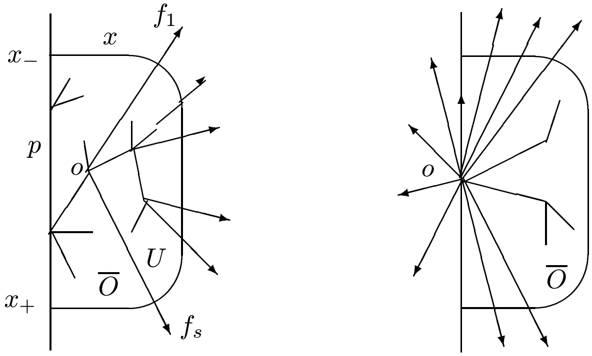
Lemma 3.7. *Suppose that a* Γ*-transversal curve* x *lies entirely in the domain* $O(p)$ *corresponding to an oval* p *of an* l*-graph* Γ*, with the exception of points* x_- *and* x_+ *lying on* p*. Then either* (1) x *intersects some edge* f *of* Γ *twice in a row in such a way that the domain bounded by* x *and* f *contains no vertices of the graph* Γ*, or* (2) x *envelops some interior vertex of* Γ*.*

Proof. Assume that (1) is not the case. Let $\overline{O}$ be the closed domain in the plane that is bounded by the curve x and p and does not contain the exterior vertex v_0.

Assume first that there exists a sequence of edges $f_1, \dots, f_s$ of Γ intersecting successively the curve x such that f_1 and f_s go out of a vertex o of Γ lying in $\overline{O}$, but not all edges in this sequence are incident to o (which we call below a $\star$-vertex). We choose this vertex o so that the domain bounded by f_1, f_s, and x, has the smallest number of vertices.

We consider the auxiliary graph $\Gamma(o)$ with vertices o, v_0, and all the vertices belonging to the domain U bounded by f_1, f_s and the arc of x lying between these edges (Fig. 1a). There exist such vertices by our choice of o. We also change the orientation (towards the vertex v_0) of all edges in $\Gamma(o)$ that go out of o and are directed inside U.

We observe that the sector bounded by the piece of x lying between adjacent edges incident to some vertex in U contains no other vertices of Γ. Otherwise we obtain in this sector an l-graph that has at most one exterior edge by our choice of the vertex o, but in contradiction with Lemma 3.2. Hence, if x does not envelop any vertex of the l-graph $\Gamma(o)$, then the number of exterior edges from an arbitrary vertex of this graph (distinct from v_0) is at most one-half of its degree plus one. This contradicts Lemma 3.4 as applied to $\Gamma(o)$ since $l \geqslant 26$, as mentioned in the definition of ovals.



a b

Figure 1

The only remaining case is a graph without $\star$-vertices.

We now include in the new graph $\Gamma(O)$ the exterior vertex v_0 and all vertices lying in $\overline{O}$. We keep all old edges of $\overline{O}$ and replace each outward edge from $\overline{O}$ by an edge with end-point v_0. It is easy to see that $\Gamma(O)$ is also an l-graph since $\overline{O}$ has at least one vertex.

By Lemma 3.2 there exists a vertex o of degree $d \geqslant l$ in the graph $\Gamma(O)$ that has at least $d-3$ consecutive exterior edges (and there are no vertices of $\Gamma(O)$ between these edges). It is clear from the definition of an oval that there are at most $\frac{1}{2}(d-2)-4$ issuing from o without intersecting the path x. Hence the remaining edges (of which there are at least $\frac{1}{2}d+2$) intersect the arc x successively (not as in Fig. 1b) because o is not a $\star$-vertex. Thus, o is enveloped by the curve x.

We consider now an arbitrary oval p in the l-graph Γ passing through some interior vertex v. Let $v' \neq v$ be another interior vertex lying in the closure of the domain $O(p)$ corresponding to this oval and let p' be some oval through v'. If the ovals p and p' are divergent at v and v' and the corresponding Γ-tame arc x is 3-separated from p' and does not lie in the closure of the domain $O(p')$, then the oval p' is said to be *derived* from the oval p at the vertices v and v'. In this case the vertex v' does not lie on p by Lemma 3.6.

Lemma 3.8. (1) *In the above notations, $O(p)$ contains the domain $O(p')$ defined by the derived oval p'.*

(2) *Let p'' be another oval derived from p at the vertices v and v'', $v'' \neq v'$. Then the domains $O(p'')$ and $O(p')$ do not overlap.*

Proof. (1) The first assertion follows from Lemma 3.6 and the fact that x does not lie in $\overline{O(p')}$.

(2) Since (by definition) one can join the vertices v and v' by a Γ-tame arc, it follows by Lemma 3.6 that the domain $O(p')$ cannot lie in $O(p'')$. The reverse inclusion is also impossible, therefore in proving the lemma by contradiction we may assume that the ovals p' and p'' have a common interior vertex o. (See Fig. 2, although we have not excluded so far the cases of $o = v'$ or $o = v''$.)

Let us introduce an auxiliary graph $\Gamma(x, y)$ by adding (or not adding) arcs x and y to the graph Γ (in a similar way to the proof of Lemma 3.6), where x and y

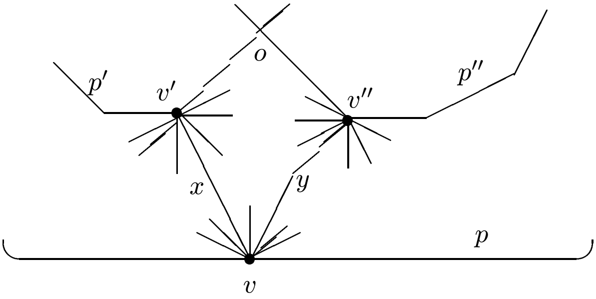


Figure 2

are Γ-tame arcs between v, v' and v, v'', respectively. As in Lemma 3.6, we consider now a simple closed path q (passing only through interior vertices of Γ) consisting of x (or the edge e if we do not include x in $\Gamma(x,y)$), y (or e'), and the pieces of the ovals p and p' from v and v', respectively, to the vertex o. In accordance with the definition of weakly curved paths p and p' and by the condition of 3-separation of the arcs x and y from p' and p'', respectively, there are only two vertices, namely, v and o, whose inner degrees relative to q can be smaller than 2. This, however, contradicts Lemma 3.1 as applied to q. We have thus proved the required second assertion of the lemma.

§ 4. List of relations

Assume that the group R in Theorem 3 has a presentation

$$R = \langle a_1, \dots, a_m \mid w = 1,\ w \in E \rangle, \tag{4.1}$$

where E is a recursively enumerable set of words on $a_1, \dots, a_m$. Adding (if necessary) new generators and relations of the form $a_j a'_j = 1$, one can assume in what follows that the set E consists only of *positive words*, that is, there are no occurences of the form a_j^{-1} in the words $w \in E$.

Based on the presentation (4.1) we now construct a series of groups following Rotman's book [6] and changing only relation (4.10).

We briefly recall (cf. [6], Chapter 12) that the positiveness of the words in E and the recursive enumerability of this set imply the existence of a deterministic one-tape Turing machine T with one head and with tape alphabet $\{s_0, s_1, \dots, s_M\}$ containing $a_1, \dots, a_m$ that stops (erasing the tape before a transition to the terminal state) if and only if the input word w (on $s_0, \dots, s_M$) belongs to the set E.

For the machine T one can construct the Markov – Post semigroup $\Gamma(T)$ with generators $s_0, \dots, s_M, h, q, q_0, \dots, q_N$, where the letter h marks the end of tape and the letters $q, q_0, \dots, q_N$ correspond to the possible states of T. This semigroup has the following property ([6], Lemma 12.4).

Lemma 4.1. *The equality $hq_1wh = q$ holds in the semigroup $\Gamma(T)$ for a word w on the alphabet $s_0, \dots, s_M$ if and only if $w \in E$.*

Let $B_0 = \langle x \rangle$ be an infinite cyclic group with generator x. The group B_1 is presented by the generators $x, h, s_0, \dots, s_M$ and the relations

$$h^{-1}xh = x^2 \quad \text{and} \quad s_\beta^{-1} x s_\beta = x^2, \quad \beta = 0, \dots, M. \tag{4.2}$$

The definition of the group B_2 and the subsequent ones depends also on the commands of the machine T. To define the group B_2 one adds to the generators of B_1 letters r_i (corresponding to the distinct commands of T) with i from I, a finite set. Further, one adds the following finite set of relations to (4.2):

$$r_i^{-1} s_\beta x r_i = s_\beta x^{-1}, \qquad i \in I, \quad \beta = 0, \dots, M, \tag{4.3}$$

$$r_i^{-1} h x r_i = h x^{-1}, \qquad i \in I, \tag{4.4}$$

$$r_i^{-1} F_i^{\#} q_{i_1} G_i r_i = H_i^{\#} q_{i_2} K_i, \qquad i \in I. \tag{4.5}$$

Here we have one relation (4.5) for each $i \in I$, where $q_{i_1}, q_{i_2} \in \{q, q_0, \dots, q_N\}$ (q_{i_1} and q_{i_2} are the states before and after the execution of the ith command); F_i, G_i, H_i, and K_i are positive (possibly empty) (s,h)-words (that is, words on the alphabet $\{s_0, \dots, s_M, h\}$); and "$V^{\#}$" denotes the result of rewriting the word V by replacing each letter by its inverse. In addition, the total number of occurrences of s-letters in these four words cannot exceed four. In [6] the interested reader can find the list of these words as defined by the commands of the machine T. In particular, we have the relation $r_{i_0}^{-1} h^{-1} q_0 h r_{i_0} = q$, but no other relations with letter q or with subword $h^{-1} q_0 h$. The letter h occurs at most once in each of the words F_i, G_i, H_i, and K_i. It can be only the first letter in F_i and H_i and only the last letter in G_i and K_i. Moreover, the word F_i (the word G_i) begins with h (ends with h) if and only if the word H_i (K_i) has the same property.

In [6] one can find also the following fairly obvious results (Lemma 12.11(i)–(iii)).

Lemma 4.2. *Assume that Q is a free group with basis $\{q_0, \dots, q_N\}$. Then each group in the chain $B_0 \leqslant B_1 \leqslant B_1 * Q \leqslant B_2$ is an HNN-extension of the preceding group. More precisely,*

1. *B_1 is an HNN-extension with base B_0 and stable letters $h, s_0, \dots, s_M$;*
2. *B_1 and Q in fact generate their free product in B_2;*
3. *B_2 is an HNN-extension with base $B_1 * Q$ and stable letters $r_i, i \in I$.*

Lemma 4.3. *Let $i \in I$ and let H be a subgroup of the group B_2 generated either by the elements $F_i^{\#} q_{i_1} G_i$, hx, and all $s_\beta x$, $\beta = 0, \dots, M$, or by the elements $H_i^{\#} q_{i_2} K_i$, hx^{-1}, and all $s_\beta x^{-1}$, $\beta = 0, \dots, M$. Let $K = gp\{x, r_i;\ i \in I\}$. Then $H \cap K = 1$ and the subgroups H and K are freely generated by the indicated elements.*

Proof. We consider an equality $V = U$ in B_2, where V is a reduced word on x and r_i, $i \in I$, and U is a product of the generators of H and their inverses. Assume that V contains a letter r_i. Since U has no such letters, Lemma 4.2 and Britton's lemma show that V contains a subword of the form $r_i^{\pm 1} x^n r_i^{\mp 1}$, where $x^n \in H$ and $n \neq 0$. By Lemma 4.2(1),(2), there exists an endomorphism of the group $B_1 * \langle q_{i_1} \rangle$ (the case with q_{i_2} is quite similar) such that $x \mapsto 1$, $q_{i_1} \mapsto q_{i_1}$, $s_\beta \mapsto s_\beta$, and $h \mapsto h$. This homomorphism maps the element x^n into 1. However, the representation of x in terms of the generators of H is a non-trivial word (since $x^n \neq 1$) mapped into a non-trivial element since q_{i_1}, the s_β ($\beta = 0, \dots, M$), and h generate freely a free subgroup by Lemma 4.2.

The same argument shows that $V = 1$ if the word V is independent of the r_i, $i \in I$, and that H and K are free subgroups.

In addition we cite Lemmas 12.14 and 12.15 in [6].

Lemma 4.4. *Let L_1 and L_2 be reduced words on the alphabet $\{x, r_i;\ i \in I\}$ and let X, Y be reduced (s, h)-words. Then the words X and Y are positive and $Xq_jY = q$ in the semigroup $\Gamma(T)$ if $L_1X^{\#}q_jYL_2 = q$ in the group B_2 for some $q_j \in \{q, q_0, \dots, q_N\}$.*

We now fix an integer $L \geqslant 15$. ($L = 1$ in [6].) A presentation of the group B_3 can be obtained by adding the letters $t_1, \dots, t_L$ to the generators of B_2 and by adding the following relations:

$$t_j^{-1} r_i t_j = r_i, \qquad i \in I, \quad j = 1, \dots, L, \tag{4.6}$$

$$t_j^{-1} x t_j = x, \qquad j = 1, \dots, L. \tag{4.7}$$

We can now obtain a presentation of the group $B = B(T)$ in its turn if we add to the presentation of B_3 some generators $k_1, \dots, k_L$ and the following relations:

$$k_j^{-1} r_i k_j = r_i, \qquad i \in I, \quad j = 1, \dots, L, \tag{4.8}$$

$$k_j^{-1} x k_j = x, \qquad j = 1, \dots, L, \tag{4.9}$$

$$\prod_{j=1}^{L} k_j^{-1}(q^{-1} t_j q) k_j (q^{-1} t_j^{-1} q) = 1. \tag{4.10}$$

We define an *h-special word* as a word $\Sigma \equiv X_0^{\#} q_j Y_0$, where $X_0 \equiv hX$, $Y_0 \equiv Yh$, X and Y are positive words on the alphabet $\{s_0, \dots, s_M\}$, $q_j \in \{q_0, \dots, q_N\}$, and “$\equiv$” is the sign of alphabetical equality. In this case we set $\Sigma^* \equiv X_0 q_j Y_0$ by definition. The above modification of relation (4.10) does not alter the proof of sufficiency in Boon’s lemma in [6].

Lemma 4.5. *If Σ is an h-special word and $\Sigma^* = q$ in the semigroup $\Gamma(T)$, then*

$$\prod_{j=1}^{L} k_j^{-1}(\Sigma^{-1} t_j \Sigma) k_j (\Sigma^{-1} t_j^{-1} \Sigma) = 1$$

in the group B.

Our modification of relation (4.10) results below in modifications of the definition of the Aanderaa groups as compared with [6].

We now rewrite presentation (4.1) in other generators:

$$R \cong R_u = \langle u_1, \ldots, u_m \mid w_u = 1,\ w_u \in E_u \rangle, \tag{4.11}$$

where the subscript u means rewriting a word $w = w(a_1, \ldots, a_m)$ as $w(u_1, \ldots, u_m)$ after the substitutions $a_j \mapsto u_j$.

Let $B * R_u$ be the free product. We add to its natural presentation by the relations (4.2)–(4.11) generators $b_1, \ldots, b_m$ and the following relations (4.12)–(4.14) and define in this way the group B_4 (recall that each letter a_j belongs to $\{s_0, \ldots, s_M\}$):

$$b_i^{-1} u_j b_i = u_j, \qquad i, j = 1, \ldots, m; \tag{4.12}$$

$$b_i^{-1} a_j b_i = a_j, \qquad i, j = 1, \ldots, m; \tag{4.13}$$

$$b_i^{-1} h k_1 h^{-1} b_i = h k_1 h^{-1} u_i, \qquad i = 1, \ldots, m. \tag{4.14}$$

Lemma 4.6. *The subgroups $gp\{u_1, \ldots, u_m\}$ and $gp\{b_1, \ldots, b_m\}$ of the group B_4 generate their direct product.*

Proof. We consider the homomorphism of the group B_4 onto the free group with basis $(x_1, \ldots, x_m)$ such that $b_i \mapsto x_i$ and the generators of $B * R_u$ are mapped into 1. It shows that the subgroups in question have trivial intersection. Hence the assertion follows from relations (4.12).

We obtain the group B_5 by adding a new generator d and relations (4.15) and (4.16) to the presentation of the group B_4:

$$d^{-1} h k_1 h^{-1} d = h k_1 h^{-1}, \tag{4.15}$$

$$d^{-1} a_i b_i d = a_i, \qquad i = 1, \ldots, m. \tag{4.16}$$

Finally, the group B_6 is the result of the addition of a generator σ and the following relations (4.17)–(4.20) to B_5:

$$\sigma^{-1}(q_1^{-1} h t_1 h^{-1} q_1)\sigma = (q_1^{-1} h t_1 h^{-1} q_1) d, \tag{4.17}$$

$$\sigma^{-1}(q_1^{-1} h t_j h^{-1} q_1)\sigma = q_1^{-1} h t_j h^{-1} q_1, \qquad j = 2, \ldots, L, \tag{4.18}$$

$$\sigma^{-1}(h k_j h^{-1})\sigma = h k_j h^{-1}, \qquad j = 1, \ldots, L, \tag{4.19}$$

$$\sigma^{-1} a_i \sigma = a_i, \qquad i = 1, \ldots, m. \tag{4.20}$$

To prove Theorem 3 we shall show that the natural homomorphism $R_u \to B_6$ is injective (Lemma 10.1) and quasi-isometric (§ 12). We now present an analogue of Lemma 12.26 in [6].

Lemma 4.7. *The group B_6 can be presented by finitely many defining relations.*

Proof. It suffices to show that each relation $w_u = 1$ in the list (4.11) follows from relations (4.2)–(4.10), (4.12)–(4.20). Below we denote by w_b the word obtained from a word $w = w(a_1, \ldots, a_m)$ by the substitution $a_i \mapsto b_i$, $i = 1, \ldots, m$.

By Lemma 4.1 for a word $w = w(a_1, \dots, a_m) \in E$ we have the equality $hq_1wh = q$ in the semigroup $\Gamma(T)$. Hence, by Lemma 4.5 as applied to $\Sigma \equiv h^{-1}q_1wh$, we obtain

$$hk_1^{-1}h^{-1}w^{-1}q_1^{-1}ht_1h^{-1}q_1whk_1h^{-1} = hP^{-1}h^{-1}w^{-1}q_1^{-1}ht_1h^{-1}q_1w \tag{4.21}$$

in B, where

$$P \equiv \prod_{j=2}^{L} k_j^{-1}(h^{-1}w^{-1}q_1^{-1}h)t_j(h^{-1}q_1wh)k_j(h^{-1}w^{-1}q_1^{-1}h)t_j^{-1}(h^{-1}q_1wh).$$

We multiply both sides of (4.21) by σ^{-1} on the left and by σ on the right. By relations (4.18), (4.19), and (4.20) the letter σ commutes with $hk_1^{-1}h^{-1}$, $hP^{-1}h^{-1}$, and w. Hence

$$hk_1^{-1}h^{-1}w^{-1}\sigma^{-1}(q_1^{-1}ht_1h^{-1}q_1)\sigma whk_1h^{-1} = hP^{-1}h^{-1}w^{-1}\sigma^{-1}(q_1^{-1}ht_1h^{-1}q_1)\sigma w.$$

By (4.17), this yields the equality

$$hk_1^{-1}h^{-1}w^{-1}(q_1^{-1}ht_1h^{-1}q_1)dwhk_1h^{-1} = hP^{-1}h^{-1}w^{-1}(q_1^{-1}ht_1h^{-1}q_1)dw.$$

Lemma 4.5 allows us to substitute P^{-1} for

$$k_1^{-1}(h^{-1}w^{-1}q_1^{-1}h)t_1(h^{-1}q_1wh)k_1(h^{-1}w^{-1}q_1^{-1}h)t_1^{-1}(h^{-1}q_1wh),$$

which, after cancellations, yields the equality $dwhk_1h^{-1} = whk_1h^{-1}w^{-1}dw$, that is,

$$d(whk_1h^{-1}w^{-1})d^{-1} = whk_1h^{-1}w^{-1}.$$

The left-hand side of this relation can be rewritten as $ww_bhk_1h^{-1}w_b^{-1}w^{-1}$ in view of (4.16), (4.13), and (4.15). Hence $w_bhk_1h^{-1}w_b^{-1} = hk_1h^{-1}$, that is, we have the equality $(hk_1^{-1}h^{-1})w_b^{-1}(hk_1h^{-1}) = w_b^{-1}$. On the other hand

$$(hk_1^{-1}h^{-1})w_b^{-1}(hk_1h^{-1}) = w_u^{-1}w_b^{-1}$$

by (4.14) and (4.12). Hence $w_u = 1$, as required.

§ 5. Diagrams, bands, and annuli

We recall the well-known van Kampen–Lyndon topological interpretation of the derivation of consequences from defining relations in groups.

A *van Kampen diagram* over a group presentation $G = \langle x_1, \dots, x_k \mid R_1, R_2, \dots \rangle$ (or briefly, by abuse of language, over the group G) with cyclically reduced defining words $R_1, R_2, \dots$ is an oriented, connected, and simply connected planar 2-complex such that each oriented edge e of it has a *label* $\varphi(e) \equiv x_i^{\pm 1}$, $i = 1, \dots, k$, where $\varphi(e^{-1}) \equiv \varphi(e)^{-1}$. In addition, the word read from the boundary $\partial\Pi$ of an arbitrary 2-cell Π (simply a *cell* in what follows) must coincide with one of the defining words $R_1, R_2, \dots$ up to a cyclic permutation and inversion.

According to van Kampen's lemma a word $w = w(x_1, \dots, x_k)$ is equal to 1 in a group G if and only if there exists a (van Kampen) diagram Δ over G with label of the boundary (reading clockwise starting from some vertex of $\partial\Delta$) alphabetically equal to w. The details can be found in [9] or [10].

If the boundary label of a cell contains some letter $x^{\pm 1}$ or a letter from some subset of letters $X^{\pm 1}$, then we call such a cell an x-cell (respectively, an X-cell) for short. For example, each cell corresponding to one of relations (4.6), (4.7), (4.10), (4.17), or (4.18) is a t_j-cell (or, briefly, a t-cell if the value of j is not important), while (t, r)-cells correspond only to relations (4.6). If one of the letters $q, q_0, \dots, q_N$ occurs in the relation, then the corresponding cell is called a Q-cell. The cells corresponding to relations (4.11) are called (u, u)-cells to distinguish them from (u, b)-cells, which correspond to (4.12) and (4.14).

By Lemma 4.7 the group B_6 is finitely presented and the relations (4.10) could be excluded from the list of defining relations. However, it seems expedient to keep them since the possibility of the replacement of some pairs of (k, t)-cells by other cells and, in particular, by (u, u)-cells, is important in §9. In a similar way, we manage to reduce the number of σ- or r-cells if the diagrams in question contain certain configurations. Thus, it is convenient to count the number of cells of each type in diagrams separately (so that they become graded in the sense of [10], §13).

The highest rank is assigned to (k, t)-cells. Particular cases of the latter are the so-called discs defined in §7. This means that, of two diagrams, the one with the largest number of (k, t)-cells has a higher *type*.

The next in rank are (σ, t)-cells; that is, of two diagrams with the same number of (k, t)-cells (for instance, zero) the one with more (σ, t)-cells has a higher type. Next follow (r, Q)-cells. After that we calculate the total number of all other cells. Their number is taken into account in the last turn in comparisons of the types of diagrams.

Obviously, the type of a diagram can be regarded as an inductive parameter. (Strictly speaking, we can define the type as a finite sequence $\tau(\Delta) = (\tau_1, \dots)$, where τ_1 is the number of (t, k)-cells in the diagram Δ, and so on.)

A diagram Δ is said to be *minimal* if it has the smallest type among all diagrams with fixed boundary label.

Lemma 5.1. *Assume that a diagram Δ contains two cells Π and Π' such that starting in opposite directions from vertices o and o' along their boundary paths $\partial\Pi$ and $\partial\Pi'$ one reads the same word V. Assume further that there exists a path $p = o - o'$ with no self-intersections in Δ such that its label U commutes with V by virtue of relations of ranks lower than the rank r of the cells Π and Π'. Then the diagram Δ is not minimal, that is, the number of cells of rank r can be reduced by 2 while preserving the boundary label* (*and the numbers of cells of higher ranks*).

Proof. To reduce the type of the diagram one can make a cut along the path p, delete the cells Π and Π', and paste a diagram in the resulting hole that interprets a deduction of the relation $UVU^{-1}V^{-1} = 1$ and consists of cells of lower rank (Fig. 3). One can find a more careful explanation in [10], §13. (In particular, it is explained there how one can make the boundary of the hole homeomorphic to a circle by means of a prior '0-refinement' of the diagram Δ.)

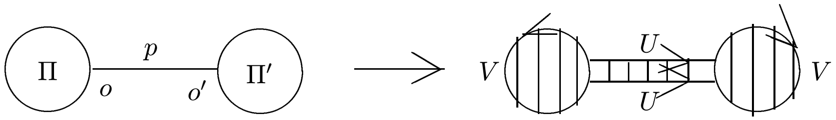


Figure 3

It is convenient to formulate some conditions and properties of diagrams using dual notions. To this end we fix a point o_Π in the interior of each cell Π and some point o_Δ in the plane outside the diagram. In a similar way we fix points o_e in the interior of each edge e of Δ. If e is an edge on $\partial\Pi$ (on $\partial\Delta$), then we fix a simple Jordan curve $l(\Pi, e)$ ($l(\Delta, e)$) connecting the vertex o_Π (o_Δ) with o_e and without other common points (apart from o_e) with the edges of Δ. We also require that $l(\Pi, e)$ and $l(\Pi, e')$ have the only common point o_Π for $e \neq e'$. A similar requirement is imposed on the curves $l(\Delta, e)$.

We consider some fixed set of letters Y (or some single letter) and some set of defining words $\mathcal{R}$ such that there is precisely one occurrence of a letter from Y with exponent 1 and precisely one with exponent -1 in each word in $\mathcal{R}$. For these sets we define a *Y-band* $S = [\pi_1, \dots, \pi_n]$ as a subdiagram built from some cells $\pi_1, \dots, \pi_n$. Namely, we require that (1) each cell $\pi_1, \dots, \pi_n$ correspond to some relation in the set $\mathcal{R}$; (2) the cells π_i and π_{i+1} be glued in S along their common edge e_i, where $e_i \neq e_{i+1}$ for $i = 1, \dots n-1$, and this edge be a *Y-edge*, that is, $\varphi(e_i) \in Y^{\pm 1}$.

Thus, the boundary path ∂S of S can by definition be represented as some product of the form $(e_0)^{-1} p_1 e_n (p_2)^{-1}$, where e_0, e_n are Y-edges of the cells π_1 and π_n, referred to as the *ends* of the band. Each *side* p_1, p_2 of the band is a product (over i) of subpaths of the boundary paths of the cells π_i enclosed between their Y-edges. The sides of a Y-band do not contain Y-edges.

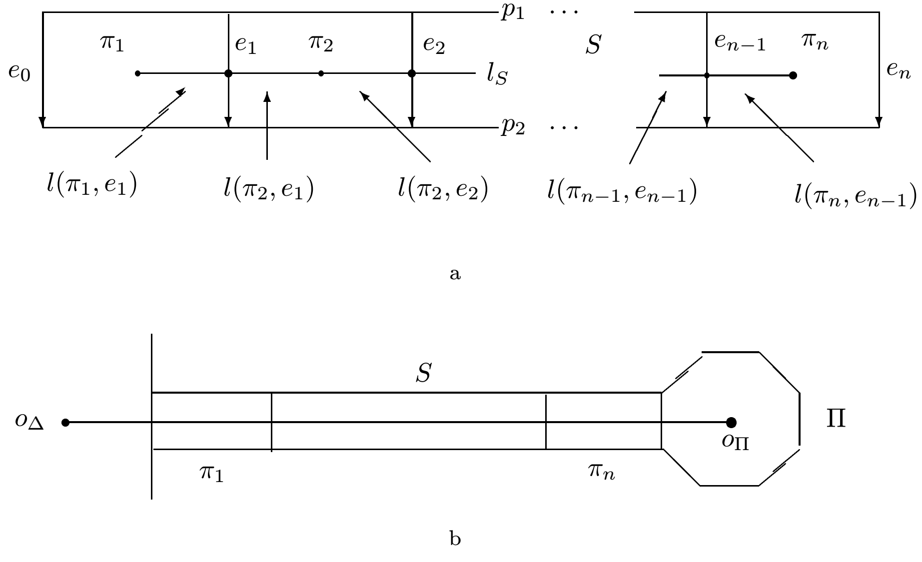


Figure 4

By definition, the *mid-line* l_S of the band S is formed by the arcs $l(\pi_1, e_1)$, $l(\pi_2, e_1)$, $l(\pi_2, e_2), \dots, l(\pi_n, e_{n-1})$ (Fig. 4a). It has no self-intersections by our choice of the relation set $\mathcal{R}$,

A Y-band of the type $[\pi_1, \dots, \pi_n, \dots, \pi_{n'}]$ is called a right extension of the band S. In a similar way we define left extensions of a band. A band that cannot be extended is said to be *maximal*. It is easy to see that a band S is maximal if either its ends coincide (that is, one has a *Y-annulus*; in this case the mid-line must also be closed), or each of its ends lies on the boundary of the diagram Δ or on the boundary of some cell Π containing a Y-edge, but not corresponding to any relation in the distinguished set $\mathcal{R}$. We say that such cells are *terminal* for Y-bands. In the last case we call the extension of the mid-line of the maximal band S beyond the ends to the points o_Δ or o_Π (Fig. 4b) the *median* of the maximal band S. One can build medians also in the case of a maximal band without cells, when $e_0 = e_n$.

If a cell π lies simultaneously in some Y-band and in some Z-band, and the pair of Y-edges separates cyclically the pair of Z-edges in $\partial\pi$, then these bands intersect by definition (their mid-lines have the common point o_π).

We now list all types of bands in diagrams over the group B_6 presented in §4 (and also over groups B_2–B_5) that will be considered in what follows.

(1) By definition, each σ-cell can be included in a σ-band (σ-annulus) of a diagram Δ over the group B_6. They correspond to relations (4.17)–(4.20). There are no terminal cells for σ-bands.

(2) By definition, a d-band (d-annulus) is made of (d, k_1)-cells and (d, a_i)-cells (see (4.15) and (4.16)). Only (σ, t_1)-cells can be terminal for d-bands (see (4.17)).

(3) b_i-bands (annuli) consist of (b_i, u)-, (b_i, a_j)-, and (b_i, k_1)-cells (see (4.12)–(4.14)). Only (d, a)-cells can be terminal for b-bands (see (4.16)).

(4) A k_j-band (annulus) is made up of (k_j, r)-, (k_j, x)-, (k_j, σ)-, and (for $j = 1$) (k_1, d)-cells (see (4.8), (4.9), (4.19), and (4.15)). Only (k, t)-cells corresponding to relations (4.10) (*hubs* in the terminology of [8]) and their generalizations, discs, which will be introduced in § 7 (they are also (k, t)-cells) can be terminal for k-bands.

(5) A t_j-band (annulus) is constructed from (t_j, r)-, (t_j, x)-, and (t_j, σ)-cells (see (4.6), (4.7), (4.17), and (4.18)). Only (k, t)-cells can be terminal for t-bands.

We call the maximal k- and t-bands with medians starting (or ending) in the interior of a (k, t)-cell (that is, of a hub or, starting from § 7, a disc) k- and *t-spokes*.

(6) An r_i-band (annulus) can consist of (r_i, t)-, (r_i, k)- (r_i, Q)-, (r_i, s_β, x)-, and (r_i, h, x)-cells (see (4.6), (4.8), (4.5), (4.3), and (4.4)). There are no terminal cells for r-bands.

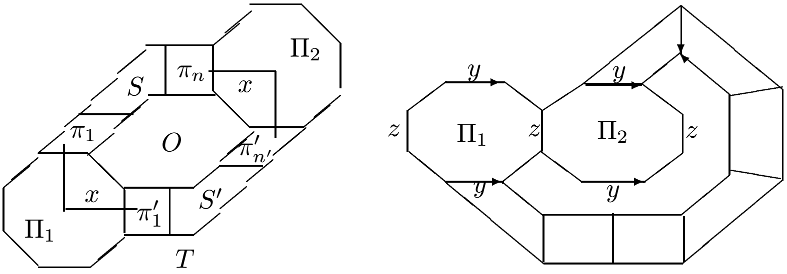
(7) We can construct a Q-band (annulus) only from (Q, r)-cells (see (4.5)). In contrast to other cases, we now use the definition of a Y-band in the case of the set $Y = \{q, q_0, \dots, q_n\}$, rather than a single letter.

Both (σ, t)-cells (4.17), (4.18) and (k, t, Q)-cells (4.10) or, starting from §7, discs (which are also (k, t, Q)-cells) can be terminal cells for Q-bands.

(8) By definition, s_β-bands and annuli (in particular, a_i-annuli) can be constructed from (s_β, x)- and (s_β, r, x)-cells (but not from (r, Q, s_β)-cells; see (4.2) and (4.3)), and if $s_\beta = a_i$, then also (a_i, σ)-, (a_i, d)-, and (a_i, b)-cells can be included (see (4.20), (4.16), and (4.13)). Only (r, Q)-cells (4.5) and discs (starting from §7) can be terminal cells for s-bands.

(9) h-bands and annuli are constructed from (h,x)- and (h,r,x)-cells (see (4.2) and (4.4)). Cells corresponding to relations (4.17)–(4.19), (4.15), (4.14), and (4.5), and discs introduced in §7 can be their terminal cells.

In addition to Y-cells, we shall encounter in what follows annuli consisting of two bands of different types. To be precise, consider now a Y-band $S = [\Pi_1, \pi_1, \dots, \pi_n, \Pi_2]$ and a Z-band $S' = [\Pi_1, \pi'_1, \dots, \pi'_{n'}, \Pi_2]$ with $Z \cap Y = \varnothing$ such that they have precisely two common cells, Π_1 and Π_2. Moreover, all four ends of these bands must lie outside the domain O bounded by the simple closed curve x formed by two mid-lines of the bands S and S' (Fig. 5a). In this case we say that the bands S and S' make up a (Y,Z)-*annulus* T with corner cells Π_1, Π_2, and with median x. The bands S and S' are called the Y- and Z-parts of the (Y,Z)-annulus T.



a b

Figure 5

Lemma 5.2. *Let* R_0 *be a cyclically reduced defining word of the form* $z_1^{\varepsilon_1} W_1 y W_2 z_2^{\varepsilon_2} W_3 y^{-1} W_4$, *where the letters* z_1, z_2, y *do not occur in the words* $W_1, \dots, W_4$, $\varepsilon_1, \varepsilon_2 = \pm 1$, *and* $z_1^{\varepsilon_1} \neq z_2^{\varepsilon_2}$. *Assume that two cells* Π_1 *and* Π_2 *in some diagram* Δ *correspond to the word* R_0 *and are corner cells in some* (Y,Z)-*annulus* (*where we have* $y \in Y$ *and* $z_1, z_2 \in Z$) *with* Z-*part consisting of just two cells,* Π_1 *and* Π_2 (*that is,* Π_1 *and* Π_2 *have a common* Z-*edge* e). *Then the diagram* Δ *is not minimal.*

Proof. We consider two possible ways of gluing the cells Π_2 to Π_1 in the diagram Δ.

In the first case some word V equal to R_0 or its cyclic permutation can be read by moving along the boundary paths of both Π_1 and, in the opposite direction, Π_2, starting with the same edge e. However, the diagram Δ is not minimal in this case by Lemma 5.1 (the word U is empty).

The second case, when the words read from $\partial\Pi_1$ and $\partial\Pi_2$ starting from e are distinct, contradicts the orientability of the Euclidean plane and the fact that all four ends of the Y- and Z-parts of a (Y,Z)-annulus lie outside the domain bounded by the median x of the annulus (that is, Fig. 5b is impossible).

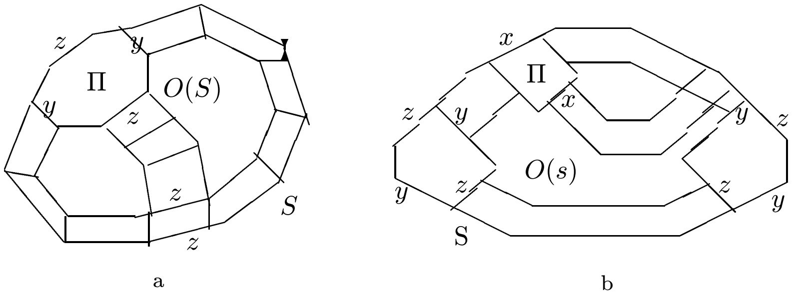
An annulus S_1 in some diagram is said to be smaller than an annulus S_2 if the domain $O(S_1)$ bounded by the median of S_1 is strictly contained in the interior of the domain $O(S_2)$ bounded by the median of S_2. The finiteness of Δ shows that the partial ordering of annuli in Δ introduced in this way is inductive.

Jordan's lemma and a consideration of the medians of annuli make the following results completely obvious.

Lemma 5.3. *Assume that a diagram Δ contains a Y-annulus S, and let $O(S)$ be the domain bounded by its median. Then we have the following results*:

(1) *if S contains a cell Π with two Y-edges separated cyclically by a pair of Z-edges and all (Y,Z)-cells in S have the same property and if, in addition, there are no Z-terminal cells in S or $O(S)$, then there exist a (Y,Z)-annulus in Δ that is smaller than the annulus S* (Fig. 6a);

(2) *if there exist no X-terminal cells in $O(S)$ and the annulus S contains no X-cells, then either there exist no X-cells in $O(S)$, or there exists an X-annulus in this diagram.*



a

b

Figure 6

Lemma 5.4. *Let S be a (Y,Z)-annulus in a diagram Δ and let $O(S)$ be the domain bounded by its median. Then we have the following results*:

(1) *if the Y-part of S contains a non-corner cell Π with two Y-edges separated cyclically by a pair of X-edges and all (X,Y)-cells in the Y-part have the same property and if, in addition, there are no X-terminal cells in S or $O(S)$, then either $Z \neq X$ and there exist X-cells in the Z-part of the annulus S, or there exists an (X,Y)-annulus in Δ that is smaller than S* (Fig. 6b);

(2) *if there exist X-terminal cells in $O(S)$ and S contains no X-cells, then either there are no X-cells in $O(S)$, or there exists an X-annulus in $O(S)$.*

§ 6. Absence of annuli in diagrams without (k,t)-cells

In a similar way to [8], the absence of annuli of various kinds in minimal diagrams without (k,t)-cells can serve as a basis for the proof of the impossibility of a multiple intersection of two bands in such diagrams. The proof of Lemma 6.1 proceeds by induction as follows: we assume that a minimal counterexample to any of its assertions has been chosen. Then all the assertions must be valid for smaller annuli, that is, there exist no smaller annuli of the indicated types. All the cross-references in the proof of the series of assertions making up Lemma 6.1 will be of this kind and we shall not repeat each time that such references are justified by the inductive hypotheses. In the considerations of bands of various kinds one should always remember the types of the cells making up these bands. To this end it is useful to have the list (1)–(9) from §5 within easy reach.

Lemma 6.1. *Let Δ be a minimal diagram without (k,t)-cells over the presentation of the group B_6 from §4, that is, Δ contains only cells corresponding to relations* (4.2)–(4.9) *and* (4.11)–(4.20)*, but not to* (4.10)*. Then Δ has no*

(1) *σ-annuli*;
(2) *r-annuli*;
(3) *(r,Q)-annuli*;
(4) *Q-annuli*;
(5) *(r,t)-annuli*;
(6) *(r,k)-annuli*;
(7) *t-annuli*;
(8) *k-annuli*;
(9) *(σ,k)-annuli*;
(10) *(σ,t)-annuli*;
(11) *(d,k)-annuli*;
(12) *(b,a)-annuli without σ- or d-cells*;
(13) *(b,k)-annuli without σ- or d-cells*;
(14) *(d,a)-annuli without t- or Q-cells*;
(15) *(σ,a)-annuli*;
(16) *(r,s)-annuli without Q-cells*;
(17) *(r,h)-annuli without Q-cells*;
(18) *d-annuli*;
(19) *b-annuli*;
(20) *a-annuli containing (d,a)-cells.*

Proof. We prove the lemma by contradiction, by choosing a minimal annulus S among all counterexamples to any of the assertions (1)–(20). Let $\Delta(S)$ be the subdiagram formed by all cells in the annulus S and all cells in the domain $O(S)$ bounded by the median of the annulus S.

(1) Let S be a σ-annulus with a (σ,k)-cell. Since in this cell the pair of σ-edges separates cyclically the pair of k-edges and a diagram without (k,t)-edges has no k-terminal cells, it follows from Lemma 5.3(1) that the diagram Δ contains some (σ,k)-annulus smaller than S. This, however, contradicts property (9) and the minimality of the chosen counterexample S. In a similar way, the existence of a (σ,t)-cell in S would mean the existence of a smaller counterexample to assertion (10).

Hence S consists only of (σ,a)-cells, and the label of the outer side of this annulus is the same as that of its inner side (see (4.20)). In this case one can identify the two sides by removing the interior of S from Δ. The diagram so obtained has a smaller type, in contradiction with the minimality of the diagram Δ.

(2) Let S be an r-annulus. The existence of a t-cell in S would contradict Lemma 5.3(1) by property (5) and the minimality of the counterexample S. Hence $\Delta(S)$ contains no t-cells by property (7) and Lemma 5.3(2), that is, there are no terminal cells for Q-bands. Hence S contains no Q-cells by Lemma 5.3(1) and

property (3), that is, no terminal cells for s- and h-bands. Consequently, there can be no (r, s_β, x)-cells or (r, h, x)-cells in S by assertions (16) and (17), that is, S contains no cells whatsoever.

(3) Let S be an (r, Q)-annulus. Then its Q-part T can consist only of r-cells, therefore T contains no non-corner cells, for otherwise one can find a smaller (r, Q)-annulus than S by Lemma 5.4. Hence the corner cells are glued together in T along a common Q-edge and they correspond to the same relation of type (4.5) since they lie in the same r_i-band. If $q_{i_1} \neq q_{i_2}$, then these cells are obviously mirror copies of each other. This contradicts the minimality of the diagram. For $q_{i_1} \equiv q_{i_2}$ we obtain a contradiction with Lemma 5.2 by setting $Y = \{r_i\}$ and $Z = Q$.

(4) Let S be a Q-annulus. Then it consists only of (r, Q)-cells, and therefore Lemma 5.3(1) and assertion (3) bring us to a contradiction with the minimality of the counterexample.

(5) Let S be an (r, t)-annulus. Then its t-part T is free of σ-cells by Lemma 5.4(1) and property (10), which holds for smaller annuli. In a similar way, property (5), which holds for smaller annuli, ensures that there are no (r, t)-cells among non-corner cells, that is, the latter can only be (t, x)-cells and, moreover, they cannot be glued mirrorwise to one another along t-edges, by Lemma 5.1. Hence each side p of the band T' constructed from T by the removal of the corner cells has the label $x^{\pm n}$, where n is the number of cells in T'.

By Lemma 5.4(2) and property (1) there are no σ-cells in S, therefore there are no σ-cells in $\Delta(S)$, that is, there are no terminal cells for Q-bands.

Again, Lemma 5.4(1) shows that the non-corner cells of the r-part of the band S cannot be Q-, t-, or k-cells by properties (3), (5), and (6), which hold for smaller annuli. Hence they can only be (r, s, x)- or (r, h, x)-cells, and after the removal of the corner cells the side $\overline{p}$ of the r-part is labelled by some word V dependent on hx and the $s_\beta x, \beta = 0, \dots, M$ (or on hx^{-1} and the $s_\beta x^{-1}$).

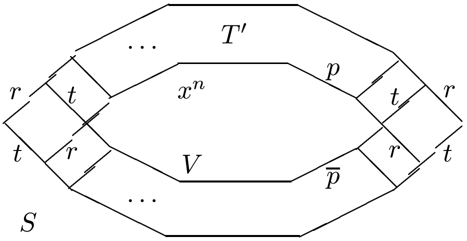


Figure 7

We now compare the labels of the sides p and $\overline{p}$ with common end-points (Fig. 7) and by van Kampen's lemma obtain

$$x^n = V(hx, s_0 x, \dots, s_M x) \tag{6.1}$$

in view of the relations corresponding to the cells lying in the domain $O(S)$. (The case of hx^{-1}, and so on, can be considered in a perfectly similar manner.)

The letters σ, t_j, and k_j do not enter equality (6.1), and there are no σ-, t- or k-annuli in $O(S)$, therefore there can be no σ-, t-, and k-cells in $O(S)$ either by properties (1), (7), and (8). In particular, there are no terminal cells for d-bands, and consequently, no d-cells by property (18). Hence there are no terminal cells for b-bands, and therefore no b-cells by property (19). Finally, $O(S)$ is free of (u,u)-cells because only (u,u) and (b,u)-cells can adjoin them.

Thus, equality (6.1) now holds in the group B_2. By Lemma 4.3 this is possible only for $n=0$. This means that the t-part of T consists only of two corner cells. Their labels must be the same since they lie in the same r_i-band. Hence Lemma 5.2 brings us to a contradiction with the minimality of the diagram Δ.

(6) Assume that S is an (r,k)-annulus. Then we can obtain a contradiction in a similar manner to the above case, replacing the references to properties (10) and (5) by references to (9) and (6). If $k=k_1$, then we rule out b-cells in the k-part of S by showing that there are no σ- or d-cells in $\Delta(S)$ and using Lemma 5.4(1) and property (13) for smaller annuli.

(7) Assume that S is a t-annulus. Then by Lemma 5.3(1) and property (10) there are no σ-cells in S, and by property (5) there are no r-cells. This ensures that S contains only (t,x)-cells and the inner and outer sides of the annulus S have the same label x^n. Hence one can identify the two sides and remove the annulus S from Δ. The resulting diagram has a smaller type, in contradiction with the minimality of Δ.

(8) Let S be a k-annulus. Then applying successively properties (9), (6), (11), (18), (13), and Lemma 5.3, we obtain that S contains no σ-, r-, d-, or b-cells, that is, it consists of (k,x)-cells. This brings us to a contradiction as in case (7).

(9) Assume that S is a (k,σ)-annulus. Then, by virtue of properties (10), (7), (2), and (6), which hold for smaller annuli, and by Lemma 5.4 there are no t- or r-cells in $\Delta(S)$. In particular, there are no terminal cells for a-bands. Thus, there are no a-cells in the σ-part by virtue of property (15) and Lemma 5.4(1). By property (9) and Lemma 5.4(1) k-cells can appear only in the k-part of the cell S. Hence there exist only corner cells in the σ-part of the annulus S. These cells have identical labels since they lie in the same k-annulus. A reference to Lemma 5.2 brings us now to a contradiction.

(10) If S is a (t,σ)-annulus, then our reasoning is quite similar to the above (with k replaced by t, and property (6) by property (5)).

(11) If S is a (d,k)-annulus, then it has no σ-cells by property (9), which holds for smaller annuli, and Lemma 5.4(1). Moreover, property (1) and Lemma 5.4(2) show that $\Delta(S)$ has no σ-cells either. Hence there are no non-corner k-cells in the d-part of the annulus S by property (11), which holds for smaller annuli, and Lemma 5.4(1).

Since $\Delta(S)$ is free of Q-terminal cells, Lemma 5.4 and property (4) guarantee that this subdiagram has no Q-cells at all; in particular, it has no a-terminal cells. By property (7) and Lemma 5.4(2) there are no t-cells in $\Delta(S)$. Lemma 5.3(1) together with properties (14) now shows that there can be no non-corner (d,a)-cells in S.

Hence there are no non-corner cells in the d-part of the annulus S. This brings us to a contradiction with Lemma 5.2 in the same standard way.

(12) Assume that S is an (a,b)-annulus without σ- and d-cells. By property (1) and Lemma 5.4(2), there are no σ-cells in the subdiagram $\Delta(S)$ either. Hence, by property (18), $\Delta(S)$ contains no d-cells; in particular, there are no terminal cells for b-bands. According to Lemma 5.4(2) and property (7) there are no t-cells

in $\Delta(S)$, that is, there are no Q-terminal cells and, therefore, no Q-cells at all, in view of property (4). Hence there are no a-terminal cells.

Now properties (12), (13), and Lemma 5.4(1) show that there are no a- or k-cells among non-corner cells of the b-part of S. There can be only (b, u)-cells.

There are no r-cells in the annulus S by Lemma 5.3(1) and property (16) since $a_j \in \{s_0, \dots, s_M\}$. Only (a, x)-cells can be non-corner cells in the a-part because cells of all other types can be ruled out in the same manner as for the b-part.

Comparing the conclusions made about the labels of the sides of the a- and b-parts, in a similar way to (5) we obtain

$$x^n = U, \tag{6.2}$$

where n is the number (or twice the number) of non-corner cells in the a-part of the annulus S and $U = U(u_1, \dots, u_m)$. Moreover, relation (6.2) holds already in the group $B_1 * R_u$ since b-cells can also be eliminated from the diagram of equality (6.2) by means of property (12). However, by Lemma 4.2 the equality (6.2) is possible in a free product only for $n = 0$.

Therefore there are no non-corner cells in the a-part of the annulus S. This brings us to a contradiction with Lemma 5.2 in the standard way.

(13) Let S be a (b, k)-annulus without σ- and d-cells. Then there are no such cells in $\Delta(S)$ either, as in case (12). The b-part of the cell S has no non-corner k-cells by Lemma 5.4(1) and property (13), which holds for smaller annuli.

By properties (6), (2), and Lemma 5.4 there are no r-cells in $\Delta(S)$, and therefore there are no a-terminal cells. Hence the b-part of S contains no a-cells by property (12).

Thus, non-corner cells of the b-part of our annulus can only be (b, u)-cells. Now, in a similar way to case (12), one obtains an equality of the form (6.2), which delivers a contradiction.

(14) Let S be a (d, a)-annulus without t- or Q-cells. By Lemma 5.4 and properties (16), (2) there are no r-cells in the subdiagram $\Delta(S)$, by properties (15) and (1) there are no σ-cells there, and by properties (11) and (8) there are no k-cells. In view of property (4) and Lemma 5.4(2) there are no Q-cells in $\Delta(S)$, that is, no a-terminal cells. Thus, by assertion (14) and Lemma 5.4(1) there are no non-corner (d, a)-cells in the d-part of the annulus S. This shows the complete absence of non-corner cells in the d-part. It remains to apply Lemma 5.2 in the standard way.

(15) Assume that S is a (σ, a)-annulus. By properties (16) and (2) for smaller annuli and by Lemma 5.4 there are no r-cells in the subdiagram $\Delta(S)$, that is, there are no a-terminal cells. Hence, by Lemma 5.4(1) and property (15) the non-corner cells of the σ-part T of the annulus S cannot be a-cells.

By properties (9), (10), and Lemma 5.4(1) k- and t-cells are also absent in S. We arrive at the conclusion that there exist only corner cells in T. Again, this contradicts Lemma 5.2.

(16) Assume that S is an (r, s)-annulus without Q-cells. The subdiagram $\Delta(S)$ contains no t-cells by Lemma 5.4 and assertions (5) and (7), and it has no k-cells by properties (6) and (8). It follows that there are no Q-cells in $\Delta(S)$ by properties (3) and (4), that is, there are no s-terminal cells. The r-part of T has no non-corner

s-cells, in view of property (16) and Lemma 5.3(1). Finally, T has no non-corner cells whatsoever. Again, this is a contradiction with Lemma 5.2.

(17) The case of the (r,h)-cell can be examined in a similar way to the previous case.

(18) Assume that S is a d-annulus. By Lemma 5.3 and property (1) there are no σ-cells in the subdiagram $\Delta(S)$, and by properties (11), (8), and Lemma 5.3 there are no k-cells in S.

By properties (7), (2), and Lemma 5.3(2), $\Delta(S)$ has no t- or Q-cells, that is, no terminal cells for a-bands. Thus, S has no (d,a)-cells by property (14). In other words, there are no cells in S at all.

(19) Assume that S is a b-annulus. The subdiagram $\Delta(S)$ has no σ-cells by Lemma 5.3(2) and property (1), that is, it has no d-terminal cells. Hence, by property (18) and the same lemma there are no d-cells in $\Delta(S)$. We can now eliminate k-cells from S using property (13).

The diagram Δ contains no r-cells by property (2) and Lemma 5.3(2), that is, it has no a-terminal cells, which enables us to rule out a-cells in S using property (12). Hence, only (b,u)-cells are possible in S, and there are identical words written on the outer and the inner sides of this annulus. This contradicts the minimality of Δ, as in case (7).

(20) Let S be an a-annulus with a (d,a)-cell. By properties (1), (15), and Lemma 5.3 there are no σ-cells in $\Delta(S)$, that is, there are no terminal cells for d-bands. Moreover, there are no t-cells in Δ by property (7). Hence, by property (4) there are no Q-cells, since there are no terminal cells for Q-bands. Thus, the presence of the (d,a)-cell in S contradicts property (14) of smaller bands.

Thus, there can be no counterexample to the assertion of Lemma 6.1.

§ 7. Diagrams over the groups B_2, C and B

Lemma 7.1. *Let* L_1 *and* L_2 *be reduced words dependent on* x *and* r_i, $i \in I$, *and let* W_1 *and* W_2 *be reduced words on* $q, q_0, \dots, q_N, s_0, \dots, s_M$, *and* h, *where there is a unique occurrence of a* Q*-letter* $q_j \in \{q, q_0, \dots, q_N\}$ *in* W_2. *Then the equality* $L_1 W_1 L_2 = W_2$ *in the group* B_2 *means that there is a unique occurrence of a* Q*-letter in the word* W_1.

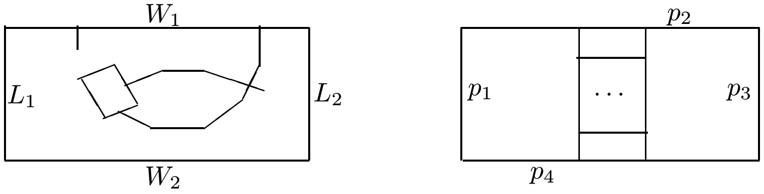
Proof. We consider a minimal diagram over B_2 with boundary $p_1p_2p_3p_4$, where $\varphi(p_1) \equiv L_1$, $\varphi(p_2) \equiv W_1$, $\varphi(p_3) \equiv L_2$, and $\varphi(p_4) \equiv W_2^{-1}$.

If the path p_2 were not simple, then the letters written on it would satisfy some non-trivial relation in contradiction to Lemma 4.2.

If p_2 has at least two Q-edges, then there must exist some Q-band S with ends lying on p_2, since the path $p_3p_4p_1$ contains only one Q-edge. In view of the above observation, this band contains at least one cell. At the same time, if there were (Q,r)-cells in it, then there would exist a (Q,r)-annulus since there are no r-edges in the path p_2 (Fig. 8a). This is a contradiction with Lemma 6.1(3), which means that there is at most one Q-edge in the path p_2.

The Q-edge of the path p_4 must be joined by a Q-band with some edge on p_2 (or merely lie on p_2, see Fig. 8b). The proof is complete.

Lemma 7.2. *Assume that a diagram* Δ *over a group* B *contains two cells,* Π_1 *and* Π_2, *satisfying relation* (4.10). *In addition, assume that these cells have common*



a b

Figure 8

adjacent spokes (*in each cell*) *connecting the points* o_{Π_1} *and* o_{Π_2} *and there are no other spokes or* (k,t)*-cells in the domain* O *bounded by these spokes. Then* Δ *is not a minimal diagram; namely, the number of its* (k,t)*-cells can be reduced by* 2 *while preserving the boundary label.*

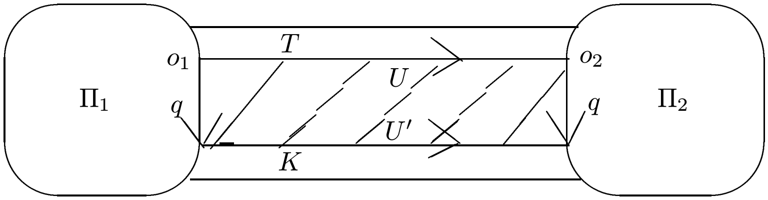


Figure 9

Proof. Thus, we have two bands: the t-band T and the k-band K with terminal cells Π_1 and Π_2, and the spokes mentioned in the statement pass through these cells (Fig. 9). Let U be the word written from the cell Π_1 up to the cell Π_2 on the side p of the band T lying in the domain O. In the same way we can define the label U' of the corresponding side of the band K. Since the spokes are chosen to be adjacent, we obtain some subdiagram (shaded in Fig. 9) with boundary label $q^{-1}Uq(U')^{-1}$. Moreover, the equality

$$U' = q^{-1}Uq \tag{7.1}$$

is valid in the group C presented by all defining relations of the group B except for (4.10), since there are no (k,t)-cells in O.

We note also that

$$Ut = tU \quad \text{and} \quad U'k = kU' \tag{7.2}$$

in the group C since all t-cells in T (all k-cells in K) correspond to commutation relations (4.6) and (4.7) (respectively, (4.8) and (4.9)).

Let V be the boundary label of the cell Π_1 (Π_2) written starting from some vertex p_- (respectively, p_+) and a Q-edge. It has the form $V \equiv V_1 \cdots V_{2L}$, where $V_i \equiv qkq^{-1}t$ for some $k = k_j^{\pm 1}$ and $t = t_l^{\pm 1}$. By (7.1) and (7.2),

$$V_iUV_i^{-1} = qkq^{-1}tUt^{-1}qk^{-1}q^{-1} = qkU'k^{-1}q^{-1} = qU'q^{-1} = U$$

in C, and therefore the word U commutes with V in the group C. Hence the diagram Δ is not minimal by Lemma 5.1.

In our further discussions we consider an auxiliary graph $\Gamma(k,t)$ constructed from the diagram Δ. By definition, its interior vertices are some points o_Π lying in the interior of the (k,t)-cells Π of Δ, and its exterior vertex is the point o_Δ. By definition, all the k- and t-spokes are edges of the graph $\Gamma(k,t)$.

Lemma 7.3. *The natural homomorphism $B_2 \to B$ is injective.*

Proof. Reasoning by contradiction we consider a minimal diagram Δ with some word (that is non-trivial in B_2) on the generators of this group written on its boundary.

Assume first that there exist (k,t)-cells in Δ and consider the graph $\Gamma(k,t)$. It contains no 1-gons since the t-edges in the (k,t)-cells are cyclically separated by k-edges. It contains no 2-gons with interior vertices either by Lemma 7.2. Hence it is a $4L$-graph and, in any case, an 8-graph in the terminology of §3 because $L > 1$. By Lemma 3.2 one can find a spoke incident to the vertex o_Δ, and therefore there exist k- and t-edges on $\partial\Delta$. However, the letters k and t are not included in the generating set of B_2. Hence Δ contains no (k,t)-cells.

In this case Δ is a minimal diagram over some group C, which is clearly an HNN-extension of the group B_2 with stable letters $t_1,\dots,k_1,\dots$. As is known, the base is always isomorphically embedded in an HNN-extension, and therefore the diagram Δ under consideration is impossible.

Lemma 12.13 in [6] (where $L = 1$) can now be modified in the following way.

Lemma 7.4. *Let Σ be an h-special word and let*

$$\prod_{j=1}^{L} k_j^{-1}\Sigma^{-1}t_j\Sigma k_j\Sigma^{-1}t_j\Sigma = 1 \tag{7.3}$$

in the group B. Then an arbitrary minimal diagram Δ for this equality contains a unique (k,t)-cell and there exist words L_1 and L_2 on the alphabet $\{x, r_i;\ i \in I\}$ such that $L_1\Sigma L_2 = q$ in B_2.

Proof. If there were no (k,t)-cells in Δ, that is, no terminal cells for k- and t-bands, then maximal k- and t-bands would intersect, as follows from the form of the left-hand side of (7.3). However, by the definition in §5 there are no cells contained simultaneously in k- and t-bands.

As noted in the proof of Lemma 7.3, $\Gamma(k,t)$ is an l-graph for $l = 4L$. Hence the number of (k,t)-cells in Δ is at most one, otherwise the total number of k- and t-edges in $\partial\Delta$ would be at least $8L-6$ by Lemma 3.4, while there are just $4L$ such edges (and $L > 1$).

The spokes of the unique (k,t)-cell pass through the $4L$ k- and t-edges in $\partial\Delta$. The 'sectors' between the spokes contain subdiagrams over the group C with boundary labels of the type $L_1\Sigma L_2 q^{-1}$ where L_1 and L_2 are some words satisfying the assertions of the lemma. This can be seen from the definition of k- and t-cells and the fact that Δ is a diagram over B. By Lemma 7.3 the equality $L_1\Sigma L_2 = q$ in the group C delivers the same equality in B_2.

Lemma 7.3 together with Lemmas 4.1, 4.4, and 4.5 gives us the following analogue of Boon's lemma (Lemma 12.20 in [6]).

Lemma 7.5. *Equality* (7.3) *holds in the group B for an h-special word Σ if and only if $\Sigma^* = q$ in the semigroup $\Gamma(T)$. In particular, a positive word w on the alphabet $\{s_0, \dots, s_M\}$ belongs to the set E* (see §4) *if and only if the word*

$$\prod_{j=1}^{L} k_j^{-1}(h^{-1}q_1wh)^{-1}t_j(h^{-1}q_1wh)k_j(h^{-1}q_1wh)^{-1}t_j^{-1}(h^{-1}q_1wh) \tag{7.4}$$

is equal to 1 *in B.*

For the 'straightening' of r- and σ-bands carried out below it is convenient to add to the defining relations of the group B (and also of B_4, B_5, and B_6) their consequences (7.3) for all h-special words Σ such that $\Sigma^* = q$ in $\Gamma(T)$; and, in particular, to add the defining words (7.4) with $w \in E$. Relations of such a kind can be found in [6] (see *Plate* 2, Lemma 12.24); in [8] their analogues are called discs.

Throughout, we call each cell in a diagram corresponding to a relation of the form (7.3) or relation (4.10) a *disc* or a (k,t)-cell. The formal definition of a minimal diagram given in §5 (although not its meaning) remains the same after the addition of discs to the list of cells that can form a diagram. One can speak of spokes going out of a disc, and so on.

Lemma 7.6. *Assume that a diagram Δ over the group B contains two discs D_1 and D_2 with common adjacent spokes and, moreover, there are no other discs or spokes in the domain O between these spokes. Then Δ is not a minimal diagram, that is, one can reduce the number of discs in it by* 2 *while preserving the boundary label.*

Proof. By Lemma 7.4 we can replace each disc by one standard (k,t)-cell corresponding to relation (4.10) and by cells of smaller ranks. The (k,t)-cells Π_1 and Π_2 obtained as modifications of D_1 and D_2 satisfy the hypothesis of Lemma 7.2, which gives us the assertion of Lemma 7.6.

Lemma 7.7. *Let $\Sigma \equiv q$ or let $\Sigma \equiv h^{-1}X^{\#}q_jYh$ be an h-special word containing the word $F_i^{\#}q_{i_1}G_i$ for some $i \in I$ or the word $H_i^{\#}q_{i_2}K_i$, that is, for some relation* (4.5) *and some words U and V we have $q_j = q_{i_1}$, $hX_i \equiv UF_i$, $Y_ih \equiv G_iV$ (respectively, $q_j = q_{i_2}$, $hX_i \equiv UH_i$, $Y_ih \equiv K_iV$). Then the word $W \equiv U^{\#}r_iH_i^{\#}q_{i_2}K_ir_i^{-1}V$ (the word $U^{\#}r_i^{-1}F_i^{\#}q_{i_1}G_ir_iV$) obtained from Σ by a single application of* (4.5) *is equal in* *a lower rank than that of relations* (4.5) *to a word of the form* $x^{n_1}r_ix^{n_2}\Sigma_1x^{n_3}r_i^{-1}x^{n_4}$ (*respectively, of the form* $x^{n_1}r_i^{-1}x^{n_2}\Sigma_1x^{n_3}r_ix^{n_4}$), *where* Σ_1 *is an* h*-special word or* $\Sigma_1 \equiv q$.

Proof. The case of $\Sigma \equiv q$ is obvious (with $n_1 = \dots = n_4$) in view of the uniqueness of the relation of the form $r_{i_0}^{-1}h^{-1}q_0hr_{i_0} = q$ in the class of relations (4.5). This is the only relation containing $h^{-1}q_0h$, which also makes the case $i = i_0$ obvious.

In every other case Σ is an h-special word, and after the application of relation (4.5) we obtain, as before, a word starting with h^{-1} and ending with h, and with no other occurrences of $h^{\pm 1}$ by the definition of relations (4.5).

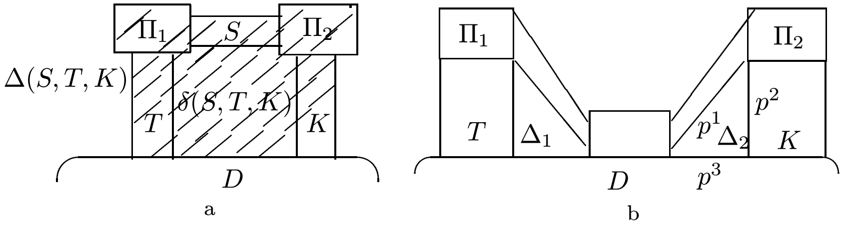
We shall consider only the first case in what follows. For a positive word T on $s_0, \dots, s_M, h$ we denote by T_x (by T_{-x}) the result of the replacement of each letter s_j, h by s_jx, hx (by s_jx^{-1}, hx^{-1}).

By relations (4.2) and since the word V is positive, the subword V of W is equal to a word of the form $V_xx^{n_4}$ for some exponent n_4. After this substitution in W we replace the subword $r_i^{-1}V_x$ by the equal (in view of (4.3)) word $V_{-x}r_i^{-1}$. Further, using relations (4.2) we can replace the subword V_{-x} by a word of the type Vx^{n_3}. Again, this is possible because the word V is positive.

After a similar transformation of the subword of the word W to the left of the letter q_{i_2} we obtain the required result.

For brevity we call an arbitrary occurrence in the word Σ having the form as in the assertion of Lemma 7.7 for some $i \in I$ an *i-occurrence.*

We consider now the following auxiliary construction. Let D be a disc in some minimal diagram Δ over the group B, and let S be some r_i-band in Δ intersecting successively a t-band T and a k-band K defined by adjacent t- and k-spokes of the disc D which, in their turn, both intersect the mid-line of the band S at a unique point, o_{Π_1} or o_{Π_2} (Fig. 10a). Let $\Delta(S,T,K)$ be the subdiagram consisting of the cells in the bands T and K (going from the disc D up to the cells Π_1 and Π_2), the cells in the band S (from Π_1 up to Π_2), and all cells lying entirely in the domain $O(S,T,K)$ bounded by the above-mentioned spokes and the mid-line of S (this diagram is shaded in Fig. 10a).



a b

Figure 10

Lemma 7.8. *If the subdiagram* $\Delta(S,T,K)$ *contains no discs and all its r-cells lie in the band* S, *then some* (r_i,Q)*-cell of the band* S *has a common* Q*-edge with the boundary segment* p *of the disc* D *lying between the ends of the bands* T *and* K (Fig. 10b), *and there is an i-occurrence in the label* $\Sigma \equiv \varphi(p)$ *of the segment* p.

Proof. By the definition of a disc, $\Sigma \equiv q$ or the word Σ is h-special. Furthermore, for any diagram over the group B the cells of the bands T and K cannot contain

Q-edges. Hence the path p has a unique Q-edge e, and there exists a cell of the band S attached to this edge in $\Delta(S,T,K)$.

Let $\delta(S,T,K)$ be the maximal subdiagram consisting of all cells in the domain $O(S,T,K)$ (it can be constructed from $\Delta(S,T,K)$ by removing the cells lying in the bands S,T, and K). Note that there are no t- or k-cells among the cells of the r_i-band $[\Pi_1,\dots,\Pi_2]$ except for its first and last cells; for otherwise, contrary to Lemma 6.1(5),(6), there would exist a (t,r_i)- or (k,r_i)-annulus. Hence the boundary label of the diagram $\delta(S,T,K)$ is a word over the generating set of the group B_2, and we can then regard this subdiagram (by Lemma 7.3) as a subdiagram over B_2.

Bearing in mind the definitions of t- and k-bands we conclude that Lemma 7.1 is applicable to the words written on $\partial\delta(S,T,K)$, and therefore the boundary of this subdiagram has no Q-edges other than $e^{\pm 1}$. After the removal of this edge the diagram $\delta(S,T,K)$ splits into two parts, Δ_1 and Δ_2 without Q-edges in their contours. There are no r-edges in their contours either, because all r-cells of the diagram $\Delta(S,T,K)$ occur in the band S. Hence, by Lemma 4.2, Δ_1 and Δ_2 can be regarded as diagrams over the group B_1.

For instance, consider the subdiagram Δ_2. It has the boundary $p^1p^2p^3$, where p^1 occurs in some side of the band S, p^2 occurs in some side of the band K, and p^3 is a subpath of p^{-1}. By the definition of a disc, $\varphi((p^3)^{-1})$ is a positive word on $s_0,\dots,s_M,h$, and by the definition of a k-band, $\varphi(p^2)$ is a word on x.

The word $U \equiv \varphi(p^1)$ is freely equal to a product $G_iV_1\dots V_n$, where $V_1,\dots,V_n$ have the form $(s_\beta x)^{\pm 1}$ or $(hx)^{\pm 1}$. (The case with K_i can be considered in a similar way.) By the minimality of the band S, in reducing U to normal form in the HNN-extension B_1 one could encounter cancellations of stable s-letters only at the junction of G_i and V_1. But the product G_iV_1 contains in this case just one letter $x^{\pm 1}$, and this product is reduced in B_1 since the word G_i is positive.

Thus, the word G_i is the beginning of the reduced (in B_1) form of the word U. By Britton's lemma the word $\varphi((p^3)^{-1})$ (which differs from U in B_1 only by a right-hand factor from $\langle x\rangle$) has the same beginning.

After a similar consideration of the subdiagram Δ_1 one obtains the required i-occurrence, which proves the lemma.

We conclude this section by a discussion of a more complicated configuration in some minimal diagram Δ over the group B.

Let D be a disc and let S be an r-band that intersects successively and without repetitions t- and k-bands $S_1,\dots,S_n$ defined by cyclically indexed spokes from D. If for any adjacent bands $T = S_j$ and $K = S_{j+1}$ (or $K = S_j$ and $T = S_{j+1}$) in this system the diagram $\Delta(S,T,K)$ contains no discs and $n > 2L+1$, then we say that the band S is *curved by the disc D*.

Lemma 7.9. *A minimal diagram Δ over the group B has no r-bands curved by discs.*

Proof. Assume that there exists such an r-band in S and suppose first that the domain O bounded by the mid-line of the band S and the spokes with indices 1 and n contains r-cells. Then we can construct from them another curved r-band, since there can be no r-annuli or (r,t)- or (r,k)-annuli in O by Lemma 6.1 (assertions (2), (5), and (6)). Using an inductive argument (on the number of r-cells in O) we can assume that all diagrams $\Delta(S, S_j, S_{j+1})$ ($j = 1, \dots, n-1$) satisfy the assumptions of Lemma 7.8.

By Lemma 7.8 one can distinguish a subdiagram $\Delta(S)$ of Δ consisting of a disc D and $(n-1)$ (r_i, Q)-cells π_j attached to D along Q-edges in every interval between the ends of the bands $S_1, \dots, S_n$ (Fig. 11a). By definitions (7.3) and (4.10) of the boundary label of a disc, all cells $\pi_1, \dots, \pi_{n-1}$ correspond to the same relation of the form (4.5); moreover, the cells lying in the same r_i-band are attached in a uniform manner: all their r-edges are directed either away from D or towards D.

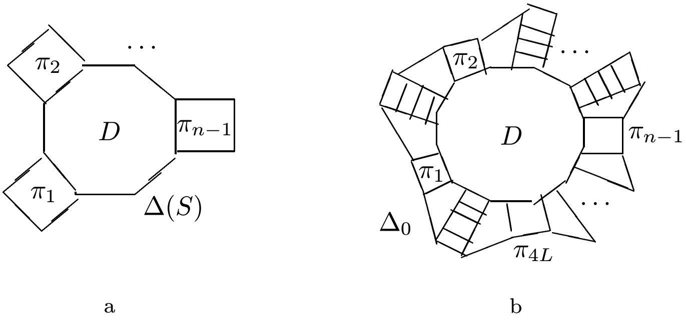


Figure 11

We now cut $\Delta(S)$ out from Δ and enlarge it as follows. First, we attach (r_i, Q)-cells to those $4L - n + 1$ Q-edges of the disc D that are free (Fig. 11b). Then we attach $4L$ diagrams of lower ranks using Lemma 7.7. Diagrams attached to adjacent pieces between the ends of spokes in ∂D are mirror copies of each other since adjacent sections have mutually inverse labels. Hence the resulting pieces of the boundary with labels dependent on r_i and x are pairwise separated by k- and t-edges of the disc D and labelled by mutually inverse words. Using relations (4.6)–(4.9) these sections can be pasted by $4L$ diagrams (shaded in Fig. 11b) so that the boundary label of the resulting diagram Δ_0 is a word of the form (7.3), but with the word Σ_1 from Lemma 7.7 in place of Σ.

It follows from the relation so obtained of the form (7.3) and Lemma 7.5 that $\Sigma_1 = q$ in the subgroup $\Gamma(T)$, that is, some disc D_1 has the same boundary label as the diagram Δ_0. Since Δ_0 was constructed by the addition to $\Delta(S)$ of some diagram (which we denote by Δ') consisting of $4L - n + 1$ (r, Q)-cells and cells of lower ranks, the diagram $\overline{\Delta}(S)$ with the same boundary label as $\Delta(S)$ can be obtained by attaching a mirror copy of the diagram Δ' to the disc D_1. The type of the diagram $\overline{\Delta}(S)$ is lower than that of $\Delta(S)$ (it has the same number of discs, 1, but fewer (r, Q)-cells since $4L - n + 1 < n - 1$). Hence the subdiagram $\Delta(S)$ of Δ, and therefore the entire diagram Δ, is not minimal.

§ 8. Absence of enveloping r-bands in diagrams over B_6

Our inclusion of extra relations in the list of defining relations and the rank assigned to (σ, t)-cells in §5 make it possible to avoid enveloping for r- and σ-bands in diagrams not only over B, but also over the group B_6.

We consider an analogue of the construction of $\Delta(S,T,K)$ before Lemma 7.8. Let D be a disc in a minimal diagram Δ over the group B_6 defined by relations (4.2)–(4.20) in §4. Let S be a σ- or r-band that intersects successively and without repetitions a t-band T_1, a k-band K, and a t-band T_2 defined by three adjacent spokes going out of the disc D. Let $\Delta(S,T_1,T_2)$ be the subdiagram consisting of the cells of the bands S, T_1, and T_2 (more precisely, their subbands isolated by the cells Π_1 and Π_2 lying at the intersection of the band S with T_1 or T_2, respectively, as in §7) and all the cells lying entirely in the domain $O(S,T_1,T_2)$ bounded by the two indicated t-spokes and the mid-line of the band S.

Lemma 8.1. *If there are no discs or r-cells in the subdiagram $\Delta(S,T_1,T_2)$ and if all its σ-cells lie in the σ-band S, then $\Delta(S,T_1,T_2)$ contains just one cell from each of the bands T_1 and T_2, namely, the cells Π_1 and Π_2, which, therefore, have common boundary t-edges with the disc D. Furthermore, if the boundary label of the disc D is the left-hand side of the equality* (7.3) *with h-special word $\Sigma \equiv h^{-1}X^{\#}q_jYh$, then $q_j \equiv q_1$ and X is the empty word, that is, the boundary label of the disc D has the form* (7.4) *with $w \equiv Y \in E$ by Lemma* 7.5*; the disc D cannot correspond to relation* (4.10)*.*

Proof. The subdiagram $\Delta(S,T_1,T_2)$ is the union of two subdiagrams, $\Delta_1 = \Delta(S,T_1,K)$ and $\Delta_2 = \Delta(S,T_2,K)$ with common cells in the band K. As is obvious from the assumptions of the lemma, (σ,t)-cells can occur in Δ_1 only in the band S. If there really were such cells distinct from Π_1, then we would necessarily obtain (σ,t)-annuli, since the boundary t-edges in the diagram $\delta(S,T_1,K)$ defined as in the proof of Lemma 7.8 can lie only on its common boundary with S. However, such annuli are impossible by Lemma 6.1(10).

Further, we note that in passing along the boundary of the subdiagram $\delta(S,T_1,K)$ we meet precisely two Q-edges: one on the boundary of the disc D, where the word Σ or the letter q is written, and another on the boundary of the (σ,t)-cell Π_1, since there are no other (σ,t)- or (r,Q)-cells in Δ_1. These two edges must simply be inverses of each other because there are no Q-cells in $\delta(S,T_1,K)$. Hence $q_j \equiv q_1$. Deleting the edge with this label we split the diagram $\delta(S,T_1,K)$ into two. The word written on the contour of one of them is $h^{-1}(X^{\#})^{-1}hx^{\pm l}$, where l is the number of (t,x)-cells in the band T_1 lying between the disc D and the cell Π_1 (see (4.7), (4.17), and (4.18)).

To prove the lemma for the cell Π_1 we note that $l = 0$. This follows from Lemma 4.2 in the case when the equality $h^{-1}(X^{\#})^{-1}hx^{\pm l} = 1$ holds not only in the group B_6, but also in B_2. Indeed, we know from the assumptions of that lemma that (k,t)-cells are not involved in the deduction of this relation. On the other hand the group defined by all the other defining relations of the group B_6 admits an obvious retraction onto the group B_2 taking the generators σ, d, b_j, u_j, k_i, and t_i for $j = 1,\dots,M$ and $i = 1,\dots,L$ to 1. Finally, the word $X^{\#}$, which is equal to $hx^{\pm l}h^{-1}$ in the group B_2, must be empty by Lemma 4.2.

By considering the subdiagram Δ_2 we prove the lemma for Π_2.

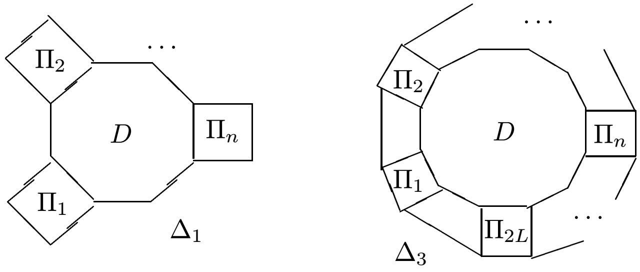
Lemma 8.2. *For arbitrary $w \in E$ the equality between $(h^{-1}q_1wh)k_1(h^{-1}q_1wh)^{-1}$ and $(h^{-1}q_1dwh)k_1(h^{-1}q_1dwh)^{-1}$ in the group B_6 can be deduced from the relations of this group presented in* §4 *without reference to (k,t)-relations* (4.10) *and (σ,t)-relations* (4.17), (4.18).

Proof. By relations (4.16) and (4.13) we obtain that $dw = ww_bd$, while $d(hk_1h^{-1})d^{-1} = hk_1h^{-1}$ by (4.15). This reduces the problem to a comparison of the words $w_bhk_1h^{-1}w_b^{-1}$ and hk_1h^{-1}, that is, to a proof of the fact that the words w_b and hk_1h^{-1} commute. However, we see from relations (4.14), (4.12), and (4.11) that $(hk_1h^{-1})^{-1}w_b(hk_1h^{-1}) = w_bw_u$, where $w_u = 1$ since $w_u \in E_u$.

Lemma 8.3. *Let D be a disc in some diagram Δ over the group B_6. Assume that the boundary of this disc has common t-edges with each of the n (σ,t)-cells $\Pi_1,\dots,\Pi_n$. Moreover, assume that these cells are attached to D in a uniform manner, that is, either all their σ-edges are directed away from D, or all their σ-edges are directed towards D. If the boundary label of the disc D has the form* (7.4) *and $n > L$, then the diagram Δ is not minimal.*

Proof. It suffices to show the non-minimality of the subdiagram Δ_1 constructed from the disc D and the cells $\Pi_1,\dots,\Pi_n$ (Fig. 12a).

We consider the diagram Δ_1 separately and make the following auxiliary constructions. First of all, we attach $2L-n$ new (σ,t)-cells to each of the $2L-n$ t-edges in ∂D that are still free (Fig. 12b). In doing that, we take care that the σ-edges of all $2L$ (σ,t)-cells in the diagram Δ_2 so obtained are directed either towards D (case A) or away from D (case B).



a b

Figure 12

Since the subpaths between t-edges of the boundary of the disc D have labels of the form $U_j^{\pm1}$, where $U_j \equiv (h_1^{-1}q_1wh)k_j^{\pm1}(h^{-1}q_1wh)^{-1}$, $w \in E$, the following word is the boundary label of the diagram Δ_2:

$$\prod_{j=1}^{L} V_jt_jV_j't_j^{-1},$$

where in case A we have

$$V_1' \equiv (h^{-1}q_1\sigma d^{-1}q_1^{-1}h)U_1(h^{-1}q_1\sigma d^{-1}q_1^{-1}h)^{-1},$$
$$V_1 \equiv (h^{-1}q_1\sigma q_1^{-1}h)U_1^{-1}(h^{-1}q_1\sigma q_1^{-1}h)^{-1}$$

and

$$V_j^{-1} \equiv V_j' \equiv (h^{-1}q_1\sigma q_1^{-1}h)U_j(h^{-1}q_1\sigma q_1^{-1}h)^{-1}$$

for $j > 1$, while in case B we have

$$V_1' \equiv (h^{-1}q_1 d\sigma^{-1}q_1^{-1}h)U_1(h^{-1}q_1 d\sigma^{-1}q_1^{-1}h)^{-1},$$
$$V_1 \equiv (h^{-1}q_1\sigma^{-1}q_1^{-1}h)U_1^{-1}(h_{-1}q_1\sigma^{-1}q_1^{-1}h)^{-1},$$

and

$$V_j^{-1} \equiv V_j' \equiv (h^{-1}q_1\sigma^{-1}q_1^{-1}h)U_j(h^{-1}q_1\sigma^{-1}q_1^{-1}h)^{-1}$$

for $j > 1$ (see relations (4.17) and (4.18)).

In case A, we can cancel out the letter d after reductions in the word V_1' on the basis of Lemma 8.2 and without reference to relations of the form (4.10), (4.17), or (4.18). If we apply after that relations (4.19) and (4.20), then the occurrences of the letter σ can also be eliminated. This brings us to the word U_1. In case B the word V_1' is also equal to U_1 in view of the relations of the group B_6 (but not using relations of the form (4.10), (4.17), or (4.18)). (In this case we get rid of σ first and of d next.) The analogous equality of each of the words V_j' to U_j for $j > 1$ can be explained more easily since no references to Lemma 8.2 are necessary in these cases.

The above arguments show that by using cells of lower ranks than those of (k, t)- and (σ, t)-cells, the diagram Δ_2 can be completed to a diagram Δ_3 with the same boundary label as that of the disc D (Fig. 12b). The diagram Δ_3 is obtained from Δ_1 by the addition of another diagram (denoted by Δ'), which consists of $2L - n$ (σ, t)-cells and cells of lower ranks. Hence we can obtain a diagram $\overline{\Delta}$ with the same boundary label as Δ_1 by attaching to D a mirror copy of Δ'. However, the type of the diagram $\overline{\Delta}$ is smaller than that of Δ_1: it contains fewer (σ, t)-cells since $2L - n < n$. Hence Δ_1 is not a minimal diagram.

Let D be a disc in a diagram Δ over the group B_6 and let S be an r- or σ-band crossing successively and without repetitions adjacent to t- and k-bands T_1, K_1, T_2, $K_2, \dots, T_n$ defined by some neighbouring spokes of the disc D. If, in addition, the subdiagrams $\Delta(S, T_j, T_{j+1})$ (defined in the beginning of this section) do not contain discs and $n > L + 1$, then we say that the band S *envelops* the disc D.

Lemma 8.4. *There can be neither r- nor σ-bands enveloping discs in a minimal diagram Δ over the group B_6.*

Proof. Reasoning by contradiction we can assume that all the r- and σ-cells of the subdiagrams $\Delta_j = \Delta(S, T_j, T_{j+1})$ lie in the band S. (This can be achieved by choosing the enveloping band 'closest' to D, as in the beginning of the proof of Lemma 7.9 with additional references to Lemma 6.1(1), (9), (10) for the elimination of σ-cells.)

First, assume that S consists of r-cells. Then there are no σ-edges on the boundary of the subdiagrams Δ_j. Hence, by Lemma 6.1(1) there are no σ-cells in these diagrams, that is, no d-terminal cells. Since $\partial\Delta_j$ has no d-edges, Δ_j has no d-cells by Lemma 6.2(18). In a similar way, b-cells can be ruled out by Lemma 6.1(19) and, as a consequence, (u, u)-cells are also eliminated. Hence Δ_j is a diagram over the group B and the band S is curved by the disc D in some subdiagram of Δ, which is defined over the group B because $2n-1>2L+1$. This, however, contradicts Lemma 7.9.

Assume now that S is a σ-band. Applying Lemma 8.1 to each of the subdiagrams Δ_j we obtain n (σ,t)-cells $\Pi_1,\dots,\Pi_n$, which are attached to the disc D in a uniform manner since they lie in the same band S. Furthermore, the boundary label of the disc D must be of the form (7.4). In this case the assertion of the lemma follows from Lemma 8.3.

§ 9. Discs and spokes in diagrams over the group B_6

This section is devoted to a trick that will help us to eliminate pairs of discs connected by several spokes not only in diagrams over the group B (of Novikov–Boon type), but also over the group B_6 (of Aanderaa type).

Lemma 9.1. *For a word $w\in E$ let* (7.4) *be the corresponding product, which we regard as a cyclic word. Let V_0 be the subword of it obtained by deleting the complementary subword of one of the forms $q_1^{-1}ht_jh^{-1}q_1whk_jh^{-1}$, where $j\neq 1$, or $hk_jh^{-1}w^{-1}q_1^{-1}ht_j^{-1}h^{-1}q_1$, where $j\neq 1$, or $q_1^{-1}ht_j^{-1}h^{-1}q_1whk_{j+1}^{-1}h^{-1}$, where $j\neq 1$* (*and the subscripts are treated modulo L*), *or $hk_j^{-1}h^{-1}w^{-1}q_1^{-1}ht_jh^{-1}q_1$, where $j\neq 1$* (*that is, the subscript of a t-letter is always different from* 1). *Then one can deduce the equality $V_0\sigma^{-1}V_0^{-1}\sigma=1$ in the group B_6 from the defining relations of this group* (*presented in* §4) *without reference to (k,t)-relations* (4.10).

Proof. The letter σ commutes with the word w in view of relations (4.20), and it commutes also with products of the form $q_1^{-1}ht_jh^{-1}q_1$ $(j\neq 1)$ or hk_jh^{-1} by (4.18) and (4.19). Hence to prove the lemma it suffices to consider, in place of V_0, its subword $V\equiv(q_1^{-1}ht_1h^{-1}q_1)whk_1h^{-1}w^{-1}(q_1^{-1}ht_1^{-1}h^{-1}q_1)$.

In view of (4.17), (4.20), and (4.19), we have

$$\sigma^{-1}V\sigma=(q_1^{-1}ht_1h^{-1}q_1d)w(hk_1h^{-1})w^{-1}(d^{-1}q_1^{-1}ht_1^{-1}h^{-1}q_1). \tag{9.1}$$

We can apply Lemma 8.2 to the subword on the right-hand side of equality (9.1) between the two occurrences of t_1. Then both occurrences of the letter d can be cancelled out and the right-hand side turns into V, as required.

We now consider an auxiliary configuration $\Delta(S,T,K)$ determined by some disc in Δ, adjacent t- and k-spokes going out of this disc, and the band S defined as before Lemma 7.8, with the only difference that S is now a σ-band (rather than an r-band) and Δ is a diagram over the group B_6 (in place of B). As in the proof of Lemma 7.8 we shall use the notation $\delta(S,T,K)$ for the subdiagram lying in the domain bounded by the two spokes and the mid-line of the band S. Let Π_1 and Π_2 be common cells of the bands T,S and K,S, respectively. We denote the subband of the band S having the form $[\Pi_1,\pi_1,\dots,\pi_n,\Pi_2]$ by S_1.

Lemma 9.2. *Assume that the above-defined subdiagram $\Delta(S,T,K)$ is minimal, contains no discs, all its σ-cells lie in the band S_1, and T is a t-band for $t\neq t_1$. Then both sides of S_1 have the same label of the form $W\equiv q_1^{-1}ht^{\pm1}h^{-1}q_1whk^{\pm1}h^{-1}$, where $t\in\{t_2,\dots,t_L\}$, $k\in\{k_1,\dots,k_L\}$, and $w\in E$.*

Proof. The boundary t- or k-edges of the subdiagram $\delta(S,T,K)$ can lie only in the boundaries of the cells $\pi_1,\dots,\pi_n$ of the band S_1. Then, however, contrary to

Lemma 6.1(10) and (9) we obtain (σ, t)- or (σ, k)-annuli. Hence the cells $\pi_1, \dots, \pi_n$ are (σ, a)-cells. Taking into account the pieces of the boundary of Π_1 with labels $(h^{-1}q_1)^{\pm 1}$ confined between t- and σ-edges, and similar pieces with labels $h^{\pm 1}$ of the boundary of Π_2, we conclude that $W \equiv q_1^{-1}ht^{\pm 1}h^{-1}q_1whk^{\pm 1}h^{-1}$, where w is some word on $a_1, \dots, a_m$. Moreover, the labels of both sides of the band S_1 are the same because all its cells correspond to commutation relations (4.18)–(4.20).

Since $\partial\delta(S_1, T, K)$ contains no σ-, d-, b-, t-, or k-edges, using successively assertions (1), (18), (19), (7), and (8) of Lemma 6.1 we conclude that there are no σ-, d-, b-, t-, or k-cells in $\delta(S, T, K)$. Hence there exist no (u, u)-cells either, that is, $\delta(S, T, K)$ is a diagram over the group B_2. This gives us an equality of the form $L'h^{-1}q_1whL'' = \Sigma$, where the words L' and L'' depend on x and r_i, $i \in I$, and $\Sigma = q$ in the semigroup $\Gamma(T)$ by the definition of the boundary label of the disc D. By Lemma 4.5 the word Σ satisfies (7.3), and therefore, by Lemma 7.4 $q = L_1\Sigma L_2 = L_1L'h^{-1}q_1whL''L_2$ in B_2 for some words L_1 and L_2 on x and the r_i, $i \in I$. Hence $hq_1wh = q$ in the semigroup $\Gamma(T)$ by Lemma 4.4, and w is a positive word. Finally, Lemma 4.1 shows that $w \in E$.

Lemma 9.3. *Assume that a diagram Δ over the group B_6 contains two discs, D_1 and D_2, with two common adjacent t- and k-spokes, where $t \neq t_1$, and the domain O between these spokes contains no discs or other spokes. Then the diagram Δ is not minimal: one can reduce the number of discs by* 2 *while preserving the boundary label.*

Proof. For a proof we can assume that there are no cells in Δ distinct from the discs D_1 and D_2, the cells of the bands T and K intersecting with the common spokes of these discs, and the cells entirely lying in the domain O. We can assume also that the subdiagram $\overline{\Delta}$ that consists of the same cells with the exception of the discs (and is shaded in Fig.13a) is minimal, since after its reduction to a minimal form its only pairs of boundary t-edges and boundary k-edges remain the ends of maximal t- and k-bands.

Further, we distinguish in $\overline{\Delta}$ maximal σ-bands $S_1, \dots, S_n$ (it is possible that $n = 0$). Since there are no (σ, t) or (σ, k)-annuli in $\overline{\Delta}$ by Lemma 6.1(9),(10) and there are no σ-edges in ∂D_1 or ∂D_2, the first and the last cells of each band $S_1, \dots, S_n$ lie in the bands T and K, respectively. Hence we can assume that the σ-bands are indexed in the direction from D_1 to D_2.

We shall prove the lemma by induction on the number n of maximal σ-bands in $\overline{\Delta}$. Note that $2n$ is the number of σ-edges in $\partial\Delta$.

If $n = 0$, then there are no σ-edges on the boundary of Δ. There are no d-edges either, since $t \neq t_1$. It easy to see that there are no b- or u-edges in $\partial\Delta$. Hence, by Lemma 6.1(1),(18),(19) there are no σ-, d-, b-, or u-cells in the diagram $\overline{\Delta}$, that is, $\overline{\Delta}$ is a diagram over the group B. The lemma follows now from Lemma 7.6.

If $n \geqslant 1$, then the σ-band S_1, together with the bands T, K and the disc D_1, form a configuration to which we can apply Lemma 9.2. Hence the sides p_1 and p_2 of S_1 (Fig. 13b) have the same label $W \equiv q_1^{-1}ht^{\pm 1}h^{-1}q_1whk^{\pm 1}h^{-1}$ for some word $w \in E$.

By Lemma 4.1 one can construct an auxiliary disc diagram consisting of a single cell D^1 with boundary label (7.4) containing the subword W. Denote by $\overline{W}$ the subword of (7.4) cyclically complementary to W. We attach to D^1 (along the

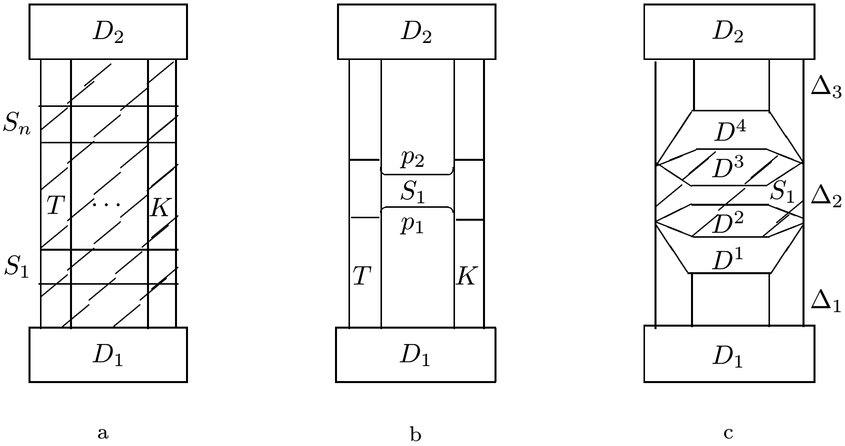


Figure 13

boundary subpaths with the same label $\overline{W}$) its mirror copy D^2. Then we obtain a diagram D^{12} with boundary label alphabetically equal to WW^{-1}.

The equality $\varphi(p_1) \equiv W$ enables us to carry out the following operation over the diagram Δ: we cut it along the path p_1 and paste D^{12} into the resulting hole (Fig. 13c). In a similar way, we can attach to the cut along the other side p_2 of the band S_1 the diagram D^{34} constructed from two discs with boundary label (7.4) that are mirror copies of each other.

One can also treat the diagram Δ' so obtained as made up of the three parts Δ_1, Δ_2, and Δ_3, the second of which consists of the discs D^2, D^3, and the band S_1 between them. The two components Δ_1 and Δ_3, which were left after our cutting Δ', contain D_1, D^1 and D^4, D_2, respectively, together with all other cells of the diagram Δ'.

We see from the definition of the diagram Δ_2 that its boundary label has the form $\overline{W}\sigma^{\pm 1}\overline{W}^{-1}\sigma^{\mp 1}$. Here the word $\overline{W}$ satisfies the assumptions of Lemma 9.1 (with $V_0 \equiv \overline{W}$) since $t \neq t_1$. Hence we can replace the subdiagram Δ_2 of Δ' by some disc-free diagram.

The boundary of Δ_1 now contains no σ-edges. Bearing in mind the above-considered case of $n = 0$, we can also replace this subdiagram of Δ' by a disc-free one. Finally, we can apply the inductive hypothesis to the subdiagram Δ_3 since its boundary contains $2(n-1)$ σ-edges.

This surgery gives us a diagram with the same boundary label as Δ, but without discs.

Let $\Gamma_t(\Delta)$ be the graph constructed as follows for an arbitrary diagram Δ over the group B_6. All points o_D in the discs are its interior vertices and o_Δ is its exterior vertex. All t-spokes are regarded as its edges. However, if two discs are joined by two t_1-spokes, then only one (any one) of them is included in $\Gamma_t(\Delta)$.

Lemma 9.4. *If the diagram Δ is minimal and contains discs, then $\Gamma_t(\Delta)$ is a $(2L-1)$-graph.*

Proof. There are no 1-gons in the graph Γ_1 because between any pair of t-spokes going out of each disc in Δ there is a k-spoke and vice versa.

Assume that a pair of spokes of Γ_t going out of D_1 and coming into D_2 forms a 2-gon. Then, clearly, all k-spokes going out of D_1 between these two also come into D_2. Hence one can find among them adjacent t- and k-spokes with $t \neq t_1$. This, however, contradicts Lemma 9.3.

Since the total number of t-spokes of each disc is $2L > 7$, this proves the lemma.

Lemma 9.5. *There can be no r-annuli in a minimal diagram over the group B_6.*

Proof. Assume that S is an r-annulus and consider a maximal subdiagram Δ lying in the domain bounded by its median. It follows from Lemma 6.1(2) that there are discs in Δ. Hence, by Lemmas 9.4 and 3.2 (and also Lemma 6.1(5),(6)) there exists a disc in the graph $\Gamma_t(\Delta)$ that is enveloped by the band S. (We recall that $2L-1-3 > L+1$.) This, however, contradicts Lemma 8.4.

§ 10. Comparison of the lengths of elements in certain subgroups

Before we carry out the comparison of lengths that is necessary for the proof of Theorem 3, we compare in the present section the lengths of elements in certain intermediate subgroups. But first of all, we prove that $R_u \cong R$ is a subgroup of B_6.

Lemma 10.1. *The natural homomorphism $R_u \to B_6$ is injective.*

Proof. We consider a minimal diagram Δ over the group B_6 corresponding to the equality to the identity of some u-word U. By definition there are no t-edges in $\partial\Delta$. Hence the diagram Δ contains no discs by Lemmas 9.4 and 3.4 because $2L-1 > 6$. By assertions (1), (18), and (19) of Lemma 6.1 we conclude successively that there are no σ-, d-, or b-cells in Δ, that is, the word U is equal to 1 in the group $B * R_u$, and therefore also in R_u.

Thus, R_u can be regarded as a subgroup of B_6. Further, let us introduce a chain $R_u \leqslant A_1 \leqslant A_2 \leqslant A_3 \leqslant A_4 \leqslant B_6$ with the following intermediate subgroups.

We set $A_1 = gp\{R_u, b_1, \dots, b_m\}$, $A_2 = gp\{A_1, k_1\}$, $A_3 = gp\{A_2, a_1, \dots, a_m\}$, and $A_4 = gp\{A_3, x, d, \sigma, r_i;\ i \in I\}$. Generating systems for these subgroups are chosen in a natural way: $\{u_1, \dots, u_m, b_1, \dots, b_m\}$ for the subgroup A_1, and so on.

In what follows we shall denote by $|V|_u$ the number of occurrences of the letters $u_1^{\pm1}, \dots, u_m^{\pm1}$ in the word V, that is, the length of the word that is the result of the deletion of all other letters.

Lemma 10.2. *Assume that a u-word U is not equal to a shorter u-word in B_6 and let V be a word on the generators of the subgroup A_1 that is equal to U in B_6. Then $|V|_u \geqslant |U|_u$ $(= \|U\|)$.*

Proof. Let Δ be a minimal diagram corresponding to the equality of the words U and V. As in the proof of Lemma 10.1 we can conclude that Δ contains no discs, σ-, or d-cells. Hence Δ is a diagram over the group B_4 and the statement follows from Lemma 4.6.

Further, let $|W|_{u,b}$ be the number of the occurrences of the letters $u_1^{\pm1}, \dots, u_m^{\pm1}$ and $b_1^{\pm1}, \dots, b_m^{\pm1}$ in the word W.

Lemma 10.3. *Let W be a word on the generators of the subgroup A_2 representing an element of A_1, and let V be a reduced word on the generators of A_1 such that some diagram Δ corresponding to the equality $W = V$ in B_6 has minimal type among all diagrams corresponding to the equalities $W = V'$ for all words V' on the generators of A_1. Then $|W|_{u,b} \geqslant |V|_u$.*

Proof. In a similar way to Lemmas 10.1 and 10.2, the diagram Δ contains no discs. Hence we can also eliminate σ- and d-cells using Lemma 6.1(1),(18).

We shall denote by $p_V p_W^{-1}$ the contour of the diagram Δ, where $\varphi(p_V) \equiv V$ and $\varphi(p_W) \equiv W$.

We observe first that the path p is simple. In particular, if e is an edge of p_V, then e^{-1} does not occur in p_V, for otherwise one could cut off from Δ a subdiagram bounded by a loop of the path p_V. This procedure removes from V a word equal to the identity in the group B_5 and reduces the type of the diagram Δ. However, this contradicts the assumption of the lemma for V.

No (u,b)- or (u,u)-cell of the diagram Δ can have common boundary edges with p_V since otherwise one could cut this cell out from Δ, changing the path p_V by some path $p_{V'}$ and reducing the type of the diagram, in contradiction with the assumption of the lemma concerning the word V.

Hence there are just two possibilities for each u-edge e of the path p_V:

(1) the edge e^{-1} occurs in the path p_W^{-1};

(2) the edge e belongs to the boundary of some (u,b,k_1)-cell Π (see (4.14)).

In the second case we choose in $\partial\Pi$ a b-band not adjacent to the u-edge e, and extend the b-band $S = [\Pi, \Pi_1, \dots, \Pi_n]$ as far as possible beyond this edge (Fig. 14a). By Lemma 6.1(19) the resulting band cannot be an annulus.

The band S has two ends, e_1 and e_2, where e_1 is the b-edge adjacent to e in $\partial\Pi$. Assume that the end e_2 of the band S occurs in the path p_V (Fig. 14b). Then S together with p_V bounds some maximal subdiagram $\delta(S)$ whose contour contains a k_1-edge f of $\partial\Pi$. This enables us to define a k_1-band $K = [\Pi, \pi_1, \dots]$ starting at Π and extending beyond f as far as possible. The band K must intersect the band S again since p_V has no k_1-edges. But this means that the bands S and K form a (b,k)-annulus, in contradiction with Lemma 6.1(13).

Hence the end e_2 of the band S must lie on the path p_W.

We consider now a b-band S' starting from some (u,b,k_1)-cell Π' distinct from Π and having a common boundary u-edge $e' \neq e$ with the path p_V. Note that Π cannot lie in S', since this would mean again the existence of a (b,k_1)-annulus or a b-annulus (see the possible configurations in Fig. 14c,d), in contradiction with Lemma 6.1(13). Hence the b-bands S and S' constructed in this way for distinct u-edges e and e' of p_V have distinct ends on the path p_W.

Thus, we associate distinct u-edges of p_V with distinct u-edges of p_W in case (1) and with distinct b-edges of this path in case (2). This proves the assertion of the lemma.

Lemma 10.4. *Let X be a word on the generators of the subgroup A_3 representing some element of A_2 and let W be a reduced word on the generators of the subgroup A_2 such that some diagram Δ corresponding to the equality $X = W$ in B_6 has minimal type among all diagrams of the equalities $X = W'$ for words W' in the generators of A_2. Then $|X|_{u,b} \geqslant |W|_{u,b}$.*

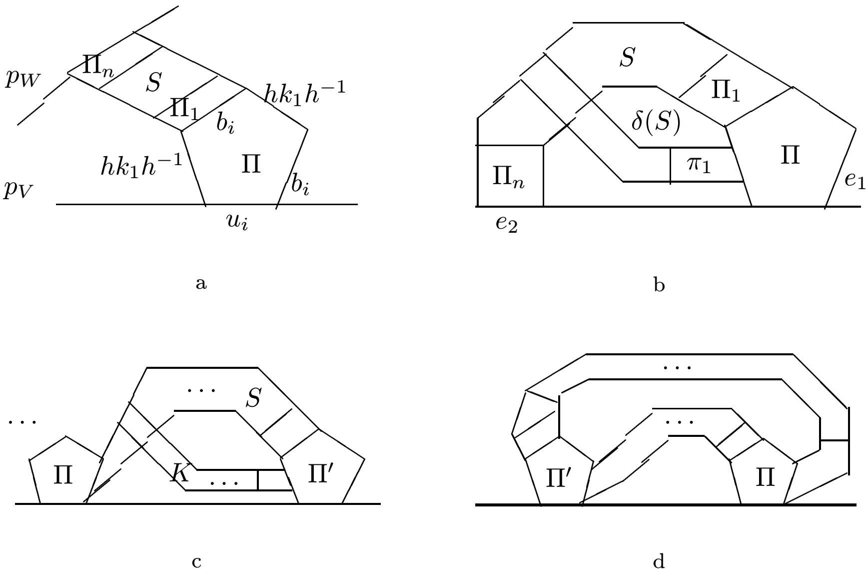


Figure 14

Proof. As in Lemmas 10.1–10.3, the diagram Δ contains no discs, σ- or d-cells. By Lemma 6.1(4) it has no Q-cells either, that is, no terminal cells for a-bands. We denote its boundary by $p_W p_X^{-1}$, where $\varphi(p_W) \equiv W$ and $\varphi(p_X) \equiv X$. As in Lemma 10.3, the path p_W is simple, but none of its u-edges e can be a boundary edge of (u,b)- or (u,u)-cells, or even of a (u,b,k_1)-cell. Hence every such edge e must also occur in the path p_X.

By the same reason, if an edge e of the path p_W is a b-edge, then there are just two possibilities:

(1) the edge e occurs in the path p_X;

(2) e belongs to the boundary of some (b,a)-cell, that is, it is an end of some maximal b-band S.

If the second end of the band S also occurs in the path p_W, then one can find a maximal subdiagram $\delta(S)$ bounded by p_W and the median of S. There exists a (b,a)-cell in S, therefore we can construct some maximal a-band T that intersects S twice and forms a (b,a)-annulus, because the path p_W contains no a-edges. This, however, contradicts Lemma 6.1(12). Consequently, in case (2) the second end of the band S is a b-edge of the path p_X.

Hence distinct b-edges of the path p_W are associated with distinct b-edges of p_X. Since we have already established such a correspondence for u-edges, the lemma is proved.

In what follows, we shall mean by $|Y|_{u,b,a}$ the number of occurrences of the letters $u_1^{\pm 1}, \dots, u_m^{\pm 1}$, $b_1^{\pm 1}, \dots, b_m^{\pm 1}$, and $a_1^{\pm 1}, \dots, a_m^{\pm 1}$ in the word Y. (In a similar way, we can define the quantity $|Y|_{u,b,s}$, which is not smaller than $|Y|_{u,b,a}$ because an a-letter is always an s-letter.)

Lemma 10.5. *Let Y be a word on the generators of A_4 representing an element of A_3, and let X be a reduced word on the generators of A_3 such that a diagram Δ*

corresponding to the equality $Y = X$ in B_6 has minimal type among all diagrams of the equalities $Y = X'$ for words X' on the generators of A_3. Then $|Y|_{u,b,a} \geqslant |X|_{u,b}$.

Proof. As in Lemmas 10.1–10.4, Δ contains no discs, and by Lemma 6.1(7) it contains no t-cells since the boundary $\partial\Delta$ has no t-edges. It follows from Lemma 6.1(4) that Δ contains no Q-cells either, for there is no Q-terminal cell in Δ and $\partial\Delta$ has no Q-edges.

The contour of the diagram Δ has the form $p_X p_Y^{-1}$, where $\varphi(p_X) \equiv X$ and $\varphi(p_Y) \equiv Y$. Just as in Lemma 10.4, the path p_X must be simple and each of its u-edges must occur in the path p_Y.

By the assumption concerning the word X, no b-edge e in the path p_X can occur in the boundary of a (b,u)-, (b,k_1)-, or (b,a)-cell because one can cut such a cell out from Δ and pass to a diagram of some equality $Y = X'$ of smaller type than Δ. Hence there exist two possibilities for e:

(1) the b-edge e of the path p_X belongs also to the path p_Y;
(2) the b-edge e lies on the boundary of a (d,a,b)-cell Π (see (4.16)).

In the second case we can carry out the same construction as in Lemma 10.3. Namely, we choose an a-edge in Π that is not adjacent to e in $\partial\Delta$ and continue the a-band $S = [\Pi, \Pi_1, \dots, \Pi_n]$ beyond this edge as far as possible (this corresponds to Fig. 14a with altered notation).

Assuming that S is an a-band we arrive at a contradiction with Lemma 6.1(20). Assuming that the end e_2 of S that is distinct from the end e_1 adjacent to e in $\partial\Delta$ lies in p_X (see Fig. 14b), we obtain an (a,d)-annulus (in the same way as a (b,k_1)-annulus appears in Lemma 10.3). This, however, contradicts Lemma 6.1(14).

Consequently, the end e_2 of S must occur in the path p_Y.

The cell Π cannot occur in an a-band that starts from another (d,a,b)-cell Π' (Fig. 14c,d) having a b-edge e' on p_X, because this would bring us either to a (d,a)-annulus again or to an a-annulus with a (d,a)-cell (in a similar way to the emergence of a (b,k_1)- or b-annulus in Lemma 10.3) since Δ contains no d-terminal cells. Hence the a-bands S and S' constructed in this way for distinct b-edges e and e' of p_X have distinct ends, e_2 and e_2', on p_Y.

Thus, our construction of an injective mapping is over: we associate the u-edges of p_X with u-edges of p_Y, and b-edges of p_X either with b-edges of p_Y (in case (1)) or with a-edges of p_Y (in case (2); these are the ends lying on the path p_Y of the a-bands S constructed for the b-edges of p_X under consideration).

§ 11. Arrangement of s-bands

The passage from the lengths of elements of the subgroup A_4 to their lengths in the group B_6 is the least trivial point in the proof of Theorem 3. The s-bands that must be considered in this case include a-bands. The corresponding estimates are based, in particular, on the properties of ovals from § 3, the hyperbolicity of the 'estimation' graph $\Gamma_t(\Delta)$ (Lemma 9.4), and the absence of r-envelopes (Lemma 8.4).

Lemma 11.1. *Let S be an s-band in a minimal diagram Δ over B_6. Then its ends e_1 and e_2 cannot lie on the boundary of the same disc D.*

Proof. Assuming the contrary we denote by $\delta(S)$ the maximal subdiagram in Δ bounded by the median of S. Its boundary has the form p_1p_2, where p_1 is a side of S and the path p_2 occurs in ∂D.

First, we assume that there exist t- or k-edges in the path p_2. Then it follows from the form (7.3) of the boundary label of a disc that t-edges alternate with k-edges in p_2. Hence there exist discs in $\delta(S)$, because there can be no t- or k-edges in p_1 and a t-band cannot intersect a k-band. We consider now the graph $\Gamma_t(\delta(S))$, which must be a $(2L-1)$-graph by Lemma 9.4. By Lemma 3.2 there exists a vertex o of this graph with at least $2L-4>2$ spokes going out of it, that is, towards the disc D. Moreover, there are no other discs of Δ between these spokes. However, $o=o_{D'}$ for some disc D', and we obtain a contradiction with Lemma 9.4 (as applied to Δ this time). Hence p_2 has no t- or k-edges, and there exist no discs in the subdiagram $\delta(S)$.

We now assume that the ends e_1 and e_2 are separated in p_2 by a Q-edge. There can exist only one such edge, as seen from (7.3) and the definition of the h-special word Σ. Since there exists no Q-edge in p_1, the boundary $\partial\delta(S)$ contains a unique Q-edge. But this is impossible in view of the absence of Q-terminal cells in $\delta(S)$.

It remains to examine the case when $\varphi(p_2)$ is a subword of the word $(X^{\#})^{-1}$ or Y^{-1}, where $\Sigma \equiv h^{-1}X^{\#}q_jYh$. Here the words X and Y are positive by the definition of a disc. Hence the fact that the plane is oriented prevents both ends of the band S from lying simultaneously on ∂D.

We shall call a subpath of the boundary of a disc labelled by an s-word an *s-segment.*

Lemma 11.2. *Let $S_1,\dots,S_n$ be maximal s-bands (not necessarily containing cells) such that their ends $e_1,\dots e_n$ lie on the same s-segment of a disc D in a minimal diagram Δ over the group B_6 (or on the same subpath p of $\partial\Delta$ containing no Q- or t-edges), while their other ends $e'_1,\dots,e'_n$ lie on the boundaries of (r,Q)-cells occurring in the same r-band S* (Fig. 15a). *Then $n\leqslant 8$.*

Proof. To prove the lemma it suffices to consider a shorter band $S=[\Pi_1,\dots,\Pi_l]$ by assuming that both cells Π_1 and Π_l are terminal (r,Q)-cells for some of the above s-bands, say, S_1 and S_2. Since there are at most four s-edges in the boundary of each (r,Q)-cell, it suffices to refute the conjecture that the band S contains an (r,Q)-cell $\Pi=\Pi_j$, where $j\neq 1,l$. We claim that this conjecture means the existence of an (r,Q)-annulus in a disc-free subdiagram, in contradiction with Lemma 6.1(3).

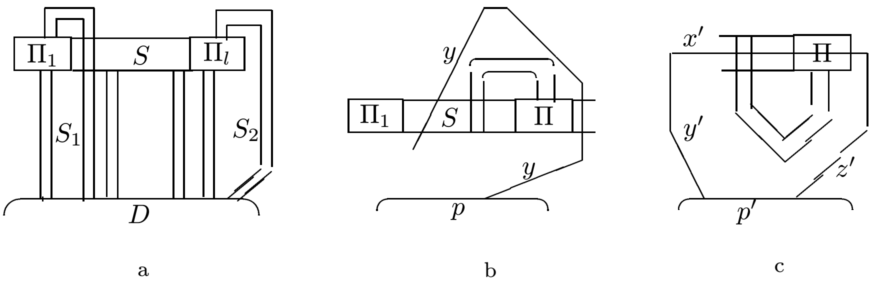


Figure 15

We consider an arbitrary simple closed curve $c = y'x'z'p'$ consisting of pieces of the medians y and z of the bands S_1 and S_2, respectively, a piece of the mid-line x of S, and a piece of the s-section p of the disc D. (There can be several such curves if the mid-lines of the bands intersect at several points.) If the maximal subdiagram $\delta(c)$ lying in c contains discs, then the r-band S must envelop some disc by Lemmas 9.4 and 3.2, because $2L-4 > L+1$, there are no t-cells in S_1 and S_2, and there are no t-edges in p. This, however, contradicts Lemma 8.4.

We recall that the (r, Q)-cell Π cannot occur in s-bands. Assume that the curve y intersects the mid-line x on different sides of Π (Fig. 15b). Then the Q-band starting from Π must intersect the r-band S again, for its mid-line must be disjoint from y. We obtain an (r, Q)-band.

Consequently, one can draw a simple closed curve $y'x'z'p'$ such that its x-part x' passes across the cell Π (Fig. 15c). Then, as above, the cell Π is a corner cell in some (r, Q)-band since the mid-line of a Q-band cannot intersect the arcs y, z, or p. This completes the proof.

For an arbitrary word W let $|W|_s$ and $|W|_r$ be the numbers of occurrences of the letters $s_0^{\pm 1}, \dots, s_m^{\pm 1}$ and of the letters $r_i^{\pm 1}, i \in I$, in W, respectively. We also set by definition

$$|W|_{s,r} = |W|_s + 60|W|_r. \tag{11.1}$$

If a minimal diagram Δ over B_6 contains discs, then $\Gamma_t(\Delta)$ is a $(2L-1)$-graph by Lemma 9.4. We now examine some oval p passing through the vertex o_D lying inside the disc D. Recall that, as shown in §3 on the basis of Lemma 3.5, an oval has precisely two exterior edges and, as a simple closed curve, it distinguishes some domain $O(p)$ in the plane. Its exterior edges, that is, the t-spokes of p, cut out from the boundary $\partial\Delta$ a subpath $\overline{p} = \overline{p}(D, p)$ that lies entirely inside $O(p)$. (It does not contain the two t-edges of Δ intersecting with the exterior spoke-edges of the oval.) We shall call $\overline{p}$ the *shadow* of the disc D on $\partial\Delta$ defined by the oval p.

By the definition of a disc, the boundary label of D is the left-hand side of equality (7.3) for some word $\Sigma(D) = \Sigma \equiv h^{-1}X^{\#}q_jYh$ (or $\Sigma \equiv q$), where $\Sigma^* = q$ in the semigroup $\Gamma(T)$. By Σ-arcs we shall mean the subpaths of the contour of a disc D that have the label $\Sigma^{\pm 1}$. We now present the key result of this section.

Lemma 11.3. *Let W be a word written on the shadow $\overline{p}$ of a disc D in a minimal diagram Δ over the group B_6. Then we have $3|\Sigma|_{s,r} = 3|\Sigma|_s \leqslant |W|_{s,r}$ for the word $\Sigma = \Sigma(D)$.*

Proof. We shall prove this lemma by induction on the number of vertices of the graph $\Gamma_t(\Delta)$ lying in the domain $O(p)$ corresponding to the oval p.

Since $2L-1 \geqslant 29$, it follows from the definition of an oval that there are at least $\frac{1}{2}(2L-3)+4$ spokes going out of D into the domain $O(p)$, that is, there are in $O(p)$ at least $2 \times 19 = 38$ successive Σ-arcs of the disc D. Dropping at least 4 first and 4 last Σ-arcs in these sequence we denote the 30 middle arcs by $p_1, \dots, p_{30}$. We also denote by $\Delta(p)$ the subdiagram formed by the cells intersecting p and lying entirely in $O(p)$. We point out that Δ contains no r-annuli by Lemma 9.5. We now consider two cases.

(1) The number N_r of maximal r-bands in the diagram $\Delta(p)$ is at least $\frac{1}{20}|\Sigma|_s$.

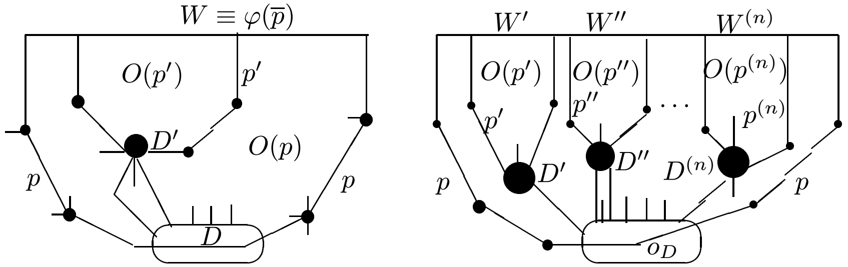
We note that at least one end of each of these bands must lie in the shadow $\overline{p}$. Indeed, otherwise we can apply Lemma 3.7 to the graph $\Gamma = \Gamma_t(\Delta)$ and to the median x of the band in question. This gives us either an (r, t)-annulus in Δ without interior discs, in contradiction with Lemma 6.1(5), or an r-band enveloping some disc, contradicting Lemma 8.4.

Hence $|W|_{s,r} \geqslant 60|W|_r \geqslant 60N_r \geqslant 3|\Sigma|_s$.

(2) $N_r < \frac{1}{20}|\Sigma|_s$.

We consider the system $\mathcal{S}$ of maximal s-bands with ends on the Σ-arcs $p_1, \dots, p_{30}$. We observe that they cannot intersect t-spokes, that is, they all lie in $\Delta(p)$. By Lemma 11.2 there are at most 8×2 bands starting from the same Σ-arc p_i and with ends on the (r, Q)-cells of the same r-band. This means that there are at most $30 \times 16 = 480$ bands of the system $\mathcal{S}$ with ends at each r-band, while by condition (2) fewer than $\frac{480}{20}|\Sigma|_s = 24|\Sigma|_s$ bands can end on all r-bands. There are at least $(30 - 24)|\Sigma|_s$ bands of the system $\mathcal{S}$ in the complementary system $\mathcal{T}$ since the second ends of the bands in $\mathcal{S}$ cannot lie on ∂D, by Lemma 11.1.

We consider now two bands S_1 and S_2 of the system $\mathcal{T}$ with ends on the same disc D' (provided that there exist such bands). The domain bounded by their medians contains no discs, for otherwise, applying Lemma 3.2 to the corresponding graph, we obtain a vertex o in the interior of one such disc that together with o_D or $o_{D'}$ makes up a 2-gon in $\Gamma_t(\Delta)$. Hence the bands S_1 and S_2 start from Σ-arcs (and end at Σ'-arcs) separated by at most one edge of the graph $\Gamma_t(\Delta)$, that is, at most two t-edges of ∂D ($\partial D'$). (Recall that one t_1-spoke of D may be erased in $\Gamma_t(\Delta)$.) This (and the inequality $2L - 1 \geqslant 29$) enables us to define the same derived oval p' passing through $o_{D'}$ using the median x of an arbitrary band of the system $\mathcal{T}$ lying between D and D' (Fig. 16a).



a

b

Figure 16

Let $p', \dots, p^{(n)}$ be all possible derived ovals constructed for the spokes ending on some discs $D', \dots, D^{(n)}$, which form a subsystem $\mathcal{T}_1$ of $\mathcal{T}$ (Fig. 16b). It follows from Lemma 3.8 that the shadows of these discs are disjoint and lie in the shadow of the disc D. Only those edges of the shadow $\overline{p}$ that do not occur in the shadows of the above-listed discs, can be terminal for the bands in $\mathcal{T}_2 = \mathcal{T} \setminus \mathcal{T}_1$. Hence, because equality (11.1) is linear, the labels $W', \dots, W^{(n)}$ of these discs' shadows satisfy the inequality

$$|W'|_{s,r} + \dots + |W^{(n)}|_{s,r} + \operatorname{card} \mathcal{T}_2 \leqslant |W|_{s,r}. \tag{11.2}$$

By the same lemma each domain $O(p'), \dots, O(p^{(n)})$ contains fewer vertices of the graph $\Gamma_t(\Delta)$ than $O(p)$. By the inductive hypothesis,

$$3|\Sigma^{(i)}|_s \leqslant |W^{(i)}|_{s,r} \tag{11.3}$$

for $i = 1, \dots, n$ and the words $\Sigma^{(i)} = \Sigma(D(i))$. At the same time,

$$6(|\Sigma'|_s + \dots + |\Sigma^{(n)}|_s) \geqslant \operatorname{card} \mathcal{T}_1 \geqslant 6|\Sigma|_s - \operatorname{card} \mathcal{T}_2, \tag{11.4}$$

since different bands in $\mathcal{T}_1$ can have ends on at most 6 Σ-arcs of each of the discs $D', \dots, D^{(n)}$. It follows now from inequalities (11.2)–(11.4) that

$$3|\Sigma|_s \leqslant \sum_{i=1}^{n} 3|\Sigma^{(i)}|_s + \frac{1}{2} \operatorname{card} \mathcal{T}_2 \leqslant \sum_{i=1}^{n} |W^{(i)}|_{s,r} + \frac{1}{2} \operatorname{card} \mathcal{T}_2 \leqslant |W|_{s,r}.$$

§ 12. Proofs of the theorems

We prove first the existence of a quasi-geodesic embedding in Theorem 3, that is, inequality (12.1) for u-words U. (Recall that by the definition of R_u and Lemma 10.1 the group R can be identified with the subgroup R_u of B_6).

Thus, let g be an arbitrary element of R_u represented by a u-word U of length $\|U\| = |U|_u = |g|_{u_1,\dots,u_m}$. For an arbitrary word Z on the generators B_6 we set $|Z|_{u,b,s,r} = |Z|_{u,b,s} + 60|Z|_r$. We claim that if Z represents the same element g, then

$$\|U\| = |U|_u \leqslant |Z|_{u,b,s,r} \leqslant 60\|Z\|. \tag{12.1}$$

To this end we fix a reduced word Y on the generators of the subgroup A_4 (see §10) such that some diagram Δ of its equality to Z in B_6 has minimal possible type. There exists such a word Y since $U = Z$ in B_6. In a similar way, for the word Y there exist words X, W, and V satisfying the assumptions of Lemmas 10.5, 10.4, and 10.3, respectively. By Lemmas 10.2–10.5 we obtain

$$|U|_u \leqslant |V|_u \leqslant |W|_{u,b} \leqslant |X|_{u,b} \leqslant |Y|_{u,b,a}.$$

Hence to prove inequality (12.1) we require the following result.

Lemma 12.1. *We have* $|Y|_{u,b,a} \leqslant |Z|_{u,b,s,r}$.

Proof. We present the contour of Δ in the form $p_Y p_Z^{-1}$, where $\varphi(Y) \equiv Y$ and $\varphi(Z) \equiv Z$. As in Lemmas 10.2–10.5, our choice of the word Y implies that p_Y is a simple path.

The boundary of no u- or b-cell of Δ can have a common edge with p_Y, for otherwise, because the boundary labels of such cells are written on the generators of A_4, one could reduce the type of the diagram by cutting these cells off and replacing Y by another word Y'. Hence each u- or b-edge of p_Y occurs also in p_Z. Consequently, to prove the lemma it suffices to establish the inequality $|Y|_a \leqslant |Z|_{s,r}$.

By the same reason as above, no edge of the path p_Y can lie on the boundary of a (σ, a)-, (b, a)-, (a, d)-, or (a, r, x)-cell. This means that there are three possibilities for the a-edges of the path p_Y: they can belong to p_Z, to the boundary of some (r, Q)-cell, or to the boundary of some disc in our diagram. We can discard the first case: removing the a-edges of the first type we shall cut the diagram Δ into several subdiagrams and prove the lemma for each of them separately, since we shall merely use the fact that Δ has minimal type.

By Lemma 9.5 there are no r-annuli in Δ. We consider now the following two cases.

(1) The number N_r of maximal r-bands in Δ is at least $\frac{1}{60}|Y|_a$.

Assume that both ends of some r-band S lie on the path p_Y. Then there are no discs in the maximal subdiagram $\delta(S)$ lying in the domain bounded by the median of this band and p_Y, as can be seen by a comparison of Lemmas 9.4, 3.2, and 8.4 since p_Y contains no t-edges. Moreover, there are no t- or k-cells in S for $k \neq k_1$, since otherwise we obtain (r, t)- or (r, k)-annuli, in contradiction with Lemma 6.1(5),(6). In a similar way, there are no Q-cells in S by Lemma 6.1(3) since there are no Q-edges in p_Y. Hence the labels of the cells in the band S are words on the generators of A_4 and we can replace Y by another word Y' by cutting out from Δ the band S together with the subdiagram $\delta(S)$. However, such a reduction of the type of the diagram contradicts our choice of the word Y.

Consequently, there are at least N_r r-edges in the path p_Z, and therefore

$$|Z|_{s,r} \geqslant 60|Z|_r \geqslant 60N_r \geqslant |Y|_a.$$

(2) $N_r < \frac{1}{60}|Y|_a$.

In this case we can deduce from Lemma 11.2 that there are at most 8 a-edges of the path p_Y that can belong to the cells of the same r-band, while the total number of a-edges of p_Y lying on the boundaries of (r, Q)-cells is less than $\frac{1}{7}|Y|_a$.

Let $\mathcal{T}$ be the set of a-edges of p_Y lying on the boundaries of some discs. We have already shown that $\operatorname{card} \mathcal{T} > \frac{6}{7}|Y|_a$.

If some edges in $\mathcal{T}$ lie on the boundary of the same disc D, then they cannot be separated in ∂D by t-edges. For the corresponding proof we need to consider a subdiagram $\delta(D)$ lying between D and p_Y. It is disc-free by Lemmas 9.4 and 3.2 since there are no t-edges in p_Y. Hence the t-band starting from some t-edge of D has no end.

Thus, the a-edges of p_Y occur in at most two Σ-arcs of D. Hence (in view of the inequality $2L - 1 \geqslant 29$) we can construct an oval p in the graph $\Gamma_t(\Delta)$ such that no a-edge p_Y lying in the disc D lies in the domain $O(p)$ defined by this oval. Then the shadow of D defined by p lies in p_Z, since there are no t-edges in the path p_Y. By Lemma 11.3 we have the following inequality for the label Z_p of this shadow:

$$|Z_p|_{s,r} \geqslant 3|\Sigma|_s \geqslant 3|\Sigma|_a \geqslant \frac{3}{2} n_D\,. \tag{12.2}$$

Here n_D is the number of a-edges of the boundary of D lying on p_Y.

The inequality $2L-1 \geqslant 29$ enables us to choose the oval p such that each Γ-tame arc x with an end-point at some common a-edge of ∂D and p_Y is 2-separated from p. Hence the ovals p_1 and p_2 constructed in this manner for two discs D_1 and D_2 that have common boundary edges, e_1 and e_2, with p_Y are divergent since the points on e_1 and e_2 can always be joined by an appropriate Γ-tame arc x. (Recall that these edges lie on the same path p_Y, which has no t-edges.)

By Lemma 3.6 the ovals p_1 and p_2 do not intersect at interior points of the graph $\Gamma_t(\Delta)$. Since the arc x lies outside the domains $O(p_1)$ and $O(p_2)$, these domains are also disjoint, which means that the shadows of the discs D_1 and D_2 are also disjoint. Hence we can add the inequalities of type (12.2) corresponding to all discs that have common boundary edges with p_Y to obtain the required relation

$$|Z|_{s,r} \geqslant \frac{3}{2}\sum n_D \geqslant \frac{3}{2}\,\frac{6}{7}\,|p_Y|_a \geqslant |Y|_a.$$

Given a group G and a function l as in Theorem 2, we can embed G in a recursively defined group R by Lemma 2.2 and Theorem 1 so that the function $g \mapsto |g|_R$ is equivalent to the function l. Inequality (12.1) proved above and Lemma 10.1 show the equivalence of the length functions in an arbitrary recursively defined finitely presented group R and in the larger group B_6 constructed from R. By Lemma 4.7, B_6 can be defined by a finite set of relations. Setting $H = B_6$ we complete the proof of Theorem 2.

To proceed from (12.1) to the inequality in Theorem 3 one can increase the finite generating set of R_u by the addition of, say, all elements of length $\leqslant 60$ on the generators $u_1,\dots,u_m$. M. V. Sapir has suggested another trick to this author.

Namely, we introduce new letters r_{ij}, where $i \in I$ and $j = 1,\dots,60$, and define a group H by the same relations as B_6 in §4, but with each generator r_i in these relations replaced by $\prod_{j=1}^{60} r_{ij}$. It is clear from this definition that we have a well-defined homomorphism $\psi\colon B_6 \to H$ (identical on all generators except for the r_i), and $\psi(r_i) = \prod_{j=1}^{60} r_{ij}$. We consider now the homomorphisms ε_j, $j = 1,\dots,60$, identical on all generators except for the r_{ik} and satisfying the relations $\varepsilon_j(r_{ij}) = r_i$ and $\varepsilon_j(r_{ik}) = 1$ for $k \neq j$. Each of them is a left inverse of ε_j. Hence ψ is an isomorphic embedding of B_6 in H.

It is obvious that the length of each element g of the group R_u (with respect to the indicated generating set $\mathcal{B}$ of H) does not exceed its length with respect to $u_1,\dots,u_m$ since $u_1,\dots,u_m \in \mathcal{B}$. On the other hand, assume that $|Z_0|_{\mathcal{B}} < |U|_u$ for some shortest word U representing an element $g \in R_u \leqslant H$ and for some word Z_0 on the alphabet $\mathcal{B}$ representing the same element. Then we choose an index j such that the total number of occurrences in Z_0 of all letters from the set $\mathcal{R}_j = \{r_{ij}\}_{i\in I}$ is at most $\frac{1}{60}$ of all occurrences of the r-letters in Z_0. Then it is easy to see that

$$|Z_0|_u + |Z_0|_b + |Z_0|_s + 60|Z_0|_j \leqslant \|Z_0\| < \|U\|, \tag{12.3}$$

where $|Z_0|_j$ is the number of r_{ij}-letters in Z_0, $i \in I$. Then, however, applying homomorphism ε_j we obtain the equality $U = Z$ in B_6 for the image Z of Z_0. Finally, from (12.3) we obtain that $|Z|_{u,b,s,r} < \|U\|$, in contradiction with (12.1).

Hence the isomorphic embedding of the group $R_u \cong R$ in the finitely presented group H constructed above preserves the lengths of elements and Theorem 3 is completely proved.

To prove Theorem 4 it suffices by Theorem 3 to show that all finitely presented groups H have the property in question. The finitely presented groups can be effectively enumerated together with their distinguished non-overlapping generating sets: $(H_1, \mathcal{A}_1)$, $(H_2, \mathcal{A}_2), \dots$. Let F be their free product with generating set $\mathcal{A} = \{a_1, a_2, \dots\}$, where $\mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$. We define a group K by adding to the natural presentation of F the generators x, y, and z and the relations

$$z^{-1}x^{-i}yx^{i}z = x^{-i}yx^{i}a_i, \qquad i = 1, 2, \dots . \tag{12.4}$$

It is easy to see that K is a finitely generated recursively presented group. By Theorem 3 it can be embedded in a finitely presented group U with the same lengths of elements. The natural homomorphisms of the groups H_j into K are injective because, by (12.4), K is an HNN-extension with basis $F * \langle x\rangle * \langle y\rangle$ and with free subgroups conjugated by z. Hence it suffices for the proof of Theorem 4 to verify that the length of each element of H_j with respect to the generators $\mathcal{A}_j$ does not exceed its length with respect to x, y, and z.

To this end we consider some word W on the generators x, y, z that represents some element of the subgroup H_j of K. Let V be a word on the generators of $\mathcal{A}_j$ such that some diagram Δ of the equality $W = V$ has the minimal possible number of cells corresponding to the relations of F and relations (12.4). It suffices to show that $\|V\| \leqslant \|W\|$.

The boundary of the diagram Δ has the form $p_V p_W^{-1}$, where $\varphi(p_V) \equiv V$ and $\varphi(p_W) \equiv W$. Here, as in Lemma 10.3, the path p_V is simple and has no common edges with the a-cells corresponding to the relations of H_j. As in Lemma 10.3, each of its edges is either included in p_W as well, or we can construct for this edge a z-band of cells of type (12.4) with an end on the path p_W. This gives us the necessary inequality $\|V\| \leqslant \|W\|$. However, for a perfect analogy with the proof of Lemma 10.3 we must explain why there are no z- or (y, z)-annuli in the minimal diagram.

This follows by the standard arguments of Lemma 6.1. In particular, they show (see the proof of properties (9) and (11) in Lemma 6.1) that a hypothetical (y, z)-annulus must consist only of two corner cells. These cells correspond to a reducible pair, since if they corresponded to distinct indices i in the list (12.4), then the subdiagram bounded by the median of the (y, z)-annulus would give us a word trivial in K with non-zero sum of exponents of x. However, such a word is non-trivial even in the retract $\langle x\rangle$ of the group K. This proves Theorem 4.

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