

On the Bass–Lubotzky Question about Quotients of Hyperbolic Groups

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We prove at Theorem 1 that any non-elementary hyperbolic group G possesses a non-trivial finitely presented quotient Q having no non-trivial subgroups of finite indices. The theorem was “commissioned” in August 1997 by H. Bass and A. Lubotzky because the statement was required to constructing of their counterexamples to Platonov’s conjecture on the representation rigid linear groups [BL]. Theorem 2 gives three equivalent reformulations of the known problem on existence of non-residually finite hyperbolic groups. © 2000 Academic Press

1. INTRODUCTION

A finitely generated group Γ is *representation rigid* if, in each dimension $n \geq 1$, Γ admits only finitely many non-isomorphic irreducible complex representations. This property is invariant under passing to a subgroup of finite index. Thus, two groups are called *commensurable* if they have isomorphic subgroups of finite indices.

Let K be a finite field extension of \mathbf{Q} , S a finite set of places containing all archimedean ones, and $K(S)$ the ring of S -integers in K . For a semi-simple linear algebraic group G over K , denote by $G(K(S))$ the group of S -integral points in $G(K)$. A group Γ is of *arithmetic type* if Γ is commensurable with a direct product of a finite set of factors $G_i(K_i(S_i))$ as defined.

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Platonov conjectured [PR] that, if Γ is a linear (over \mathbf{C}), finitely generated, representation rigid group, then Γ is of arithmetic type. The principal aim of the Bass–Lubotzky paper [BL] is to construct counter-examples to the Platonov conjecture. The desired group is an equalizer of two homomorphisms of a particular hyperbolic group G onto a non-trivial finitely presented group Q which has no proper subgroups. (See [BL] for details and more explanations.) But it was unproved that such a quotient Q does exist. Therefore the following theorem was “commissioned” from the author by Bass and Lubotzky.

THEOREM 1. *Let G be a non-elementary hyperbolic group. Then G has a non-trivial quotient $Q = G/N$ which is finitely presented and possesses no non-trivial finite homomorphic images.*

The main goal of our paper is to prove Theorem 1 because it is required by Bass and Lubotzky to complete the construction of the counter-examples for the Platonov conjecture. Our proof is strongly attached to the papers [O1, O2] where a “generalized small cancellation method” has been developed for obtaining quotients of hyperbolic groups with prescribed properties (e.g., as “Tarski monsters”). Taking the opportunity, we notice below that the results of [O1] imply the following corollary.²

THEOREM 2. *The following assertions are equivalent:*

- (1) *There exists a non-residually finite hyperbolic group.*
- (2) *There exists a non-trivial hyperbolic group having no proper subgroups of finite indices.*
- (3) *There exists a finite collection of non-elementary hyperbolic groups such that the set of the isomorphism classes of their common non-abelian, finite, simple homomorphic images is finite.*

Whether statement (1) is true or not is a well-known open problem. Gromov has conjectured [G, 5.B] that assertion (2) is valid. If every hyperbolic group is residually finite, then for every sufficiently large prime p and every integer l there is a finite non- p -group generated by some elements a, b such that any non-trivial product of the generators of length $\leq l$ has order p . (See Problem 12.64 in [KN] and the remark at the end of [IO].) Finally, notice that every hyperbolic group G is residually periodic of bounded exponent. (See [O3] for a torsion free G and [IO] for a general case.) It seems that even the task to construct a finitely presented, residually periodic (of bounded exponent) group which is not residually finite should be difficult.

²A different proof that statements (1) and (2) are equivalent was recently provided by I. Kapovich and D. Wise [KW].

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2. CONSTRUCTION FOR THE QUOTIENTS

Let us recall just one of a number of equivalent definitions of hyperbolicity.

A group G with a finite generating set A is *hyperbolic* if its Cayley graph $\Gamma = \Gamma(G, A)$ is hyperbolic; that is, for some constant $\delta = \delta(G, A)$, any geodesic triangle in Γ is δ -thin. This means that each of its sides belongs to the closed δ -neighborhood of the union of the other two sides.

A group G is *elementary* if it has a cyclic subgroup of finite index.

We use the symbol “ \equiv ” for the letter-by-letter equality of the words and the notation $\|W\|$ for the length of a word W in a fixed group alphabet. Introduce a 2-variable word

$$w(a, b) \equiv (bab^{-1})a(ba^{-1}b^{-1})a^{-2}, \tag{1}$$

and recall the Baumslag argument [B].

LEMMA 2.1 [B]. *If $w(a, b) = 1$ for some elements a, b of a finite group then $a = 1$.*

The elements a and a^2 must have the same order since they are conjugate. Thus, the order n of a is odd. The order of $c = bab^{-1}$ is also equal to n . Since $cac^{-1} = a^2$, we get $a = c^nac^{-n} = a^{2^n}$, which means $2^n \equiv 1 \pmod{n}$. Assuming that $n > 1$, we show that such a congruence is not possible even mod p , for the smallest prime divisor p of n .

Indeed, by the Fermat theorem, $2^{p-1} \equiv 1 \pmod{p}$ since $2 < p$. Therefore the order d of 2 in \mathbf{F}_p^* is less than p (and greater than 1). In view of the congruence $2^n \equiv 1 \pmod{p}$, we conclude that $d \mid n$ for $1 < d < p$, contrary to the choice of p .

Let $A = \{a_1, \dots, a_n\}$ be a system of non-trivial generators for a group G . The words A_j, B_j, C_j ($1 \leq j \leq n$) over the alphabet $A^{\pm 1}$ will be specified later. Denote by N the normal closure of the elements $w(A_j, B_j), w(A_j a_j, C_j) \in G$ ($1 \leq j \leq n$), where the word w is defined by equality (1).

LEMMA 2.2. *Suppose that G is hyperbolic and non-elementary. Then the words A_j, B_j, C_j ($1 \leq j \leq n$) can be chosen so that $N \neq G$.*

This statement is sufficient to obtaining Theorem 1. Indeed, G being a hyperbolic group, is finitely presented [G, 2.3.A]. Hence the quotient $Q = G/N$ is finitely presented. By Lemma 2.1 one has $A_j = 1$ ($1 \leq j \leq n$)

in any finite quotient of Q because the group Q satisfies relations $w(A_j, B_j) = 1$. Similarly, $A_j a_j = 1$ in each of the finite quotients of Q . Therefore $a_j = 1$ ($1 \leq j \leq n$) in any finite quotient of Q . Since the natural images of a_1, \dots, a_n generate the group Q , it has no non-trivial quotients at all.

Therefore the proof of Lemma 2.2 in Section 5 completes the proof of Theorem 1.

3. THE LONG WORDS A_j , B_j , AND C_j

It has been mentioned by Gromov [G] (see also [O1, Lemma 1.16]) that an element g of infinite order in a hyperbolic group G is contained in a unique maximal elementary subgroup $E(g) = E_G(g)$ (*elementary closure* or *elementarizer* of g).

Elements g and h of infinite order are called *commensurable* ([O1]; the notion differs from the commensurability of groups mentioned in Section 1) in a group G if $g^k = xh^l x^{-1}$ for some non-zero integers k, l and for some $x \in G$. For a hyperbolic G , this condition is equivalent to the conjugation $E(g) = xE(h)x^{-1}$.

For a subgroup H of a hyperbolic group G , its elementarizer $E(H) = E_G(H)$ is defined to be $\bigcap E(h)$, where h ranges over the elements of infinite orders in H . If the subgroup H is not elementary, $E(H)$ is the unique maximal finite subgroup of G normalized by H ([O1, Proposition 1]; notice also that $E(G)$ is the kernel of the action of G on the hyperbolic boundary ∂G induced by left multiplications on G). Therefore the quotient $G/E(G)$ has no non-trivial finite normal subgroups. Hence to prove Lemma 2 it suffices to assume that $E(G) = 1$ because the quotient of a non-elementary hyperbolic group modulo a finite normal subgroup is also a non-elementary hyperbolic group [GH, Corollary 23(ii)].

LEMMA 3.1. *Let G be a non-elementary hyperbolic group with $E(G) = 1$. Then G has infinitely many pairwise non-commensurable elements g_1, g_2, \dots such that $E(g_i) = \langle g_i \rangle$ is an infinite cyclic subgroup for every $i = 1, 2, \dots$*

The claim is just a simplification of that of Lemmas 3.4 and 3.8 in [O1] in the case $H = G$ and $E(G) = 1$.

LEMMA 3.2. *For the elements g_1, g_2, \dots given by Lemma 3.1, and for any integer $l \geq 1$, there are elements $x_{jk} \in G$ ($j = 1, 2, \dots, l; k = 1, \dots, l$) such that*

$$(E1) \quad x_{jk} \notin E(g_j) = \langle g_j \rangle;$$

$$(E3) \quad \text{if } ax_{jk} = x_{jk}b \text{ and } a, b \in \langle g_j \rangle \text{ then } a = b = 1;$$

(E4) if $a, b \in \langle g_j \rangle$ and $ax_{jk} = x_{ji}b$ then $k = i$;

(E5) the product a_jx_{j1} does not belong to the cyclic subgroup $\langle g_j \rangle$.

The elements x_{jk} , satisfying conditions (E1), (E3) and (E4), were fixed in [O1, Lemma 3.7] for every g_j . (We omit condition (E2) from [O1], since it is trivial for $H = G$ and $E(G) = 1$.) It has been proved there that elements x_{jk} can be chosen as powers g_{jk}^d of some elements g_{jk} non-commensurable with g_j , for an arbitrary sufficiently large exponent $d = d_j$. Therefore condition (E5) holds for almost every d as well, because otherwise one gets $a_jg_{j1}^{d_1}, a_jg_{j1}^{d_2} \in \langle g_j \rangle$ for $d_1 \neq d_2$, i.e., $g_{j1}^{d_1-d_2} \in \langle g_j \rangle$, and the elements g_{j1} and g_j will be commensurable.

Suppose the elements g_1, g_2, \dots are represented by some geodesic words W_1, W_2, \dots on the letters $a_1^{\pm 1}, \dots, a_n^{\pm 1}$. (A word is *geodesic* if it is not equal in G to any shorter word.) Let X_{jk} be geodesic words representing the elements x_{jk} ($j = 1, 2, \dots; k = 1, \dots, l$). Define

$$A_j = A_j(l, m) \equiv X_{j1}W_j^m X_{j2}W_j^m \cdots X_{jl}W_j^m \tag{2}$$

for $j = 1, \dots, n$. (The large integers l, m will be specified in Lemma 4.2 and in the proof of Lemma 2.2 in Section 5). Also define

$$B_{j-n} \equiv X_{j1}W_j^{m^2} X_{j2}W_j^{m^2} \cdots X_{jl}W_j^{m^2} \tag{3}$$

for $j = n + 1, \dots, 2n$. (In particular, the word B_j is much longer than A_j for large m .) Finally, set

$$C_{j-2n} \equiv X_{j1}W_j^{m^2} X_{j2}W_j^{m^2} \cdots X_{jl}W_j^{m^2} \tag{4}$$

for $j = 2n + 1, \dots, 3n$.

4. SMALL CANCELLATIONS OVER HYPERBOLIC GROUPS

Denote by $\|p\|$ the combinatorial length of a path p in the Cayley graph $\Gamma = \Gamma(G, A)$ of the non-elementary hyperbolic group G (or in a van Kampen diagram, appearing in Section 5).

For $\lambda > 0$ and $c \geq 0$, a path p is said to be (λ, c) -quasi-geodesic if for any subpath q , the distance between the initial and the terminal vertices of q is at least $\lambda\|q\| - c$. A word U is (λ, c) -quasi-geodesic if it is the label of a (λ, c) -quasi-geodesic path in Γ .

Denote by \mathcal{N} the set of all cyclically reduced non-empty words representing the elements of the normal closure of the Baumslag word $w(a, b)$ given by (1) in the free group $F(a, b)$. By the Magnus theorem on 1-related

groups [LS, Proposition II.5.1], every word of \mathcal{N} contains both letters a and b .

Consider a substitution $a \mapsto A_j, b \mapsto B_j$ or $a \mapsto A_j a_j, b \mapsto C_j$ ($j \in \{1, \dots, n\}$) for a word v from \mathcal{N} . The result will be called a *relation H-word* (because we want to use a terminology which is similar to that in [O2]). An *H-word* is a result of an analogous substitution into any reduced word v on letters $a^{\pm 1}, b^{\pm 1}$. Recall that by our definition of the words (2)–(4), they depend on parameters l and m .

LEMMA 4.1. *There exists $\lambda > 0$ such that for any $l > 0$ there exist $c > 0$ and $m_0 > 0$ such that any H-word is (λ, c) -quasi-geodesic for every $m \geq m_0$.*

Proving this lemma, we meet a new case compared with [O2, Lemma 5]. Namely, after a concatenation of a subword $A_j a_j$ to $A_j a_j$ in an *H-word*, a subword $a_j X_{j_1}$ appears at their juncture. (That was not the case in [O2].) However, the same argument as in [O2] now works as well because we have the added condition (E5) choosing elements x_{j_1} in Lemma 3.2.

Introduce the small cancellation condition we need. First define ϵ -pieces for $\epsilon \geq 0$. The definition below is similar to that in [O1].

Let R and R' be cyclic permutations of relation *H-words*. A word U is called an ϵ -piece of the word R (and of any cyclic permutation of R) with respect to G (more precisely, with respect to a finite presentation of G) if

- (1) $R \equiv UV, R' \equiv U'V'$ for some U, V, U', V' ;
- (2) $U' = YUZ$ in G for some words Y, Z where $\|Y\|, \|Z\| \leq \epsilon$;
- (3) the word $YRY^{-1}(R')^{-1}$ does not vanish in G and it not a conjugate of a relation *H-word* R'' in G .

(It was sufficient to demand in [O1, 4] that the word $YRY^{-1}(R')^{-1}$ should be just non-trivial in G .) Up to the definition of ϵ -pieces, our definition of the $C(\epsilon, \mu, c, \rho)$ -condition (for some $\epsilon \geq 0, \mu > 0, \lambda > 0, c \geq 0, \rho > 0$) coincides with that in [O1]. This means that

- (C1) $\|R\| \geq \rho$ for every relation *H-word* R ;
- (C2) every cyclic permutation of a relation *H-word* R is (λ, c) -quasi-geodesic;
- (C3) for any ϵ -piece of any *H-word* R , inequalities $\|U\|, \|U'\| < \mu\|R\|$ hold (with the preceding notation).

The following lemma is an analog of [O, Lemma 4.2, O2, Lemma 6].

LEMMA 4.2. *Let \mathcal{R} be the system of relation H-words (which depends on parameters m, l). Then there exists $\lambda > 0$ such that for any $\mu > 0$ there exist $l > 0, c \geq 0$ such that for any $\epsilon \geq 0, \rho > 0$ there exists $m_0 > 0$ such that the $C(\epsilon, \mu, \lambda, c, \rho)$ -condition holds for \mathcal{R} provided that $m \geq m_0$.*

(C1) Every relation H -word R contains one of the subwords $A_j^{\pm 1}, B_j^{\pm 1}, C_j^{\pm 1}$, and therefore its length is greater than m . Since the parameter m_0 can be chosen after $\epsilon, \mu, \lambda, c$, and ρ , we have $\|R\| \geq \rho$.

(C2) The choice of the quasi-geodesic constants $\lambda = \lambda(W_1, \dots, W_{3n})$, $c = c(l)$ is provided by Lemma 4.1.

(C3) The choice of parameters l and then m_0 , which guarantees condition (C3), is similar to that in [O2, Lemma 6], but we need an addition to the proof of Lemma 6 in [O2].

We say that two relation H -words V, V' have *the same type* if for some $j \leq n$ and $(S, T) = (A_j, B_j)$ or $(S, T) = (A_j a_j, C_j)$, one has the equalities $V \equiv v(S, T), V' \equiv v'(S, T)$ where $v, v' \in \mathcal{N}$.

To verify condition (C3) for properly chosen l and m_0 , we take two paths r and r' in the Cayley graph Γ of G , labeled by some cyclic permutations R and R' of relation H -words. Assume that their initial subpaths u and u' are labeled by words U, U' with $\|U\| \geq \mu\|R\|$, and there is a closed path labeled by the word $YUZ(U')^{-1}$ as in the definition of an ϵ -piece. We want to prove that U is not a piece.

By definition, *entire* vertices of paths r, r' correspond to the partition of the labeling H -words into subwords $A_j^{\pm 1}, B_j^{\pm 1}$ or into $(A_j a_j)^{\pm 1}, C_j^{\pm 1}$. Following the proof of Lemma 6 in [O2], we will explain that R and R' are of the same type, and will find a common vertex on r and r' which is entire in both paths r and r' . (In [O2], we had to find two entire vertices on r and r' , respectively, connected by a path labeled by a word from a subgroup $E(D)$. Now, this word is trivial, since the assumption $E(G) = 1$ from Section 2 gave us a chance to formulate a condition (E3) stronger than [O2]. Hence the entire vertices will coincide in the Cayley graph.) Then, for the cyclic permutations R_1 and R'_1 of the words R and R' , respectively, that can be read starting from the common entire vertex, their ratio $S \equiv R_1(R'_1)^{-1}$ is freely conjugate to a relation H -word R'' (or to the empty word). However, the word S is conjugate in G to the word $YRY^{-1}R'^{-1}$, and consequently, U is not a piece.

To obtain the declared properties of the words R, R' and the paths r, r' , we distinguish the only point of the proof of Lemma 6 in [O2], where an additional explanation is required now as compared with [O2]. This happens if R and R' are of different types, but for the same j , R is a cyclic permutation of some $v(A_j, B_j)$ and R' is a cyclic permutation of some $v'(A_j a_j, C_j)$ (or vice versa). Proving by contradiction as in [O2], we get a common entire vertex (if l and m_0 are large enough) on subpaths r and r' .

Notice for this case that $\|B_j\|, \|C_j\| > m/2\|A_j a_j\|$ by definitions (2), (3), and by the choice m_0 after other parameters.

The cyclic permutation R of a relation H -word must contain a subword B_j , since the letter b occurs in every word of the set \mathcal{N} . Therefore the

inequality $\|B_j\| > m/2\|A_j\|$ and our choice of m (after μ) allow us to assume that

$$\|A_j\| < \mu/10\|R\| \leq 1/10\|U\|.$$

This makes it possible to find a sufficiently long occurrence Z in U neighbor to A_j (i.e., ZA_j or A_jZ is a longer occurrence) which is a subword of either $B_j^{\pm 1}$ or A_j . Again, the first assumption can lead to a contradiction, exactly as in [O2], because, by Lemma 3.1, the element g_{j+n} , used in the definition of the word B_j , is not commensurable with the elements q_j and g_{j+2n} , used in the definitions of the words A_j and C_j . In the second case, the word R has a subword A_jA_j such that the two entire vertices of r , corresponding to the beginnings of these two occurrences of A_j , coincide in Γ with similar entire vertices of r' , corresponding to a subword $(A_ja_j)(A_ja_j)$. Then $A_ja_j = A_j$ in G , and $a_j = 1$, contrary to our choice of the generators for the group G . Thus the second case is impossible too.

5. PROOF OF THEOREMS 1 AND 2

Proof of Lemma 2.2. It remains to be proven that $N \neq G$. For this goal, by using a van Kampen diagram, we show below that $a_1 \neq 1$ in the quotient $Q = G/N$.

Unlike [LS], where an edge of a diagram can be labeled by a word, we will suppose that an edge is labeled by a letter from the generating set of G (see [O4] for details).

The group Q can be given by a finite set of defining relations for G and by all the relations having relation H -words in the left-hand side. If Δ is a diagram over this presentation, then as in [O1], the faces corresponding to the relations of G are said to be 0-faces, and the faces labeled by the relation H -words are called \mathcal{R} -faces.

Thus, the assumption that $a_1 = 1$ in Q leads to the existence of a van Kampen diagram Δ of perimeter 1 over the presentation of Q , which must contain an \mathcal{R} -face. We suppose that Δ has a minimal number of \mathcal{R} -faces among the diagrams with the same boundary label (in particular, it is reduced in the sense of [O2]).

Lemma 6.6 in [O1] (the “generalized Greendlinger lemma”) says that for any hyperbolic group G and any $\lambda > 0$, there is a $\mu > 0$ such that for any $c \geq 0$, there are $\epsilon \geq 0$ and $\rho > 0$ with the following property. If the $C(\epsilon, \mu, \lambda, c, \rho)$ -condition holds for a system of relators, imposed on the group G , then a reduced diagram Δ , which has \mathcal{R} -faces, must contain an \mathcal{R} -face Π , such that “almost the whole” boundary of Π is situated “alongside” the boundary of Δ . Now we can refer to the claim of that

lemma (also see the remark made after the proof of Lemma 7 in [O2]) because no changes are required in the proof of it except for applications of Lemmas 4.1 and 4.2 instead of their analogs from [O1, O2].

More precisely, [O1, Lemma 6.6] claims that there is an \mathcal{R} -face Π in Δ with $\partial\Pi$ labeled by a relation H -word R , which enjoys the following property. There exists such a subdiagram Δ_0 of Δ , which is a diagram over G (i.e., it has no \mathcal{R} -faces), and the boundary $\partial\Delta_0$ can be partitioned into a product $yuzu'$, where u is a subpath of $\partial\Pi$, u' is a subpath of $\partial\Delta$, and (for $\mu < 1/46$)

$$\|y\|, \|z\| \leq \epsilon, \|u\| > 1/2\|R\|. \tag{5}$$

We may assume now that the parameters $m, l, \epsilon, \mu, \lambda, c$, and ρ are taken in accordance with Lemma 4.2 and the analog of Lemma 6.1 in [O1]. Thus, all the words A_j, B_j, C_j are fixed now.

Any cyclic permutation of R is (λ, c) -quasi-geodesic, by Lemma 4.2. Hence the triangle inequality and (5) yield the inequalities

$$\|u'\| \geq \lambda\|u\| - c - 2\epsilon > 1/2\lambda\rho - c - 2\epsilon > 1$$

because the parameter ρ can be chosen large enough as compared with λ, c , and ϵ in Lemma 4.2. But $\|u'\| \leq \|\partial\Delta\| = \|a_1\| = 1$. The contradiction completes the proofs of Lemma 2.2 and Theorem 1. ■

Remark. In the same manner, the inequality $V \neq 1$ in Q can be proven for any “short” word which is non-trivial in G . In other words, a variation of the parameters provides us a system of residualizing homomorphisms of a non-elementary hyperbolic group G onto quotients like in Theorem 1.

The following simplification of Theorem 2 from [O1] will help below to prove Theorem 2 formulated in Section 1.

LEMMA 5.1. *Let H_1, \dots, H_k be non-elementary subgroups of a hyperbolic group such that $E_G(H_1) = \dots = E_G(H_k) = E(G) = 1$. Then there exists a non-elementary hyperbolic quotient \tilde{G} of G , such that the image of each subgroup H_1, \dots, H_k under the natural epimorphism $G \rightarrow \tilde{G}$ coincides with \tilde{G} . ■*

The implications (2) \Rightarrow (1), (3) are obvious.

(1) \Rightarrow (2). Let G be a non-residually finite hyperbolic group. Denote by K the non-trivial intersection of the kernels of all homomorphisms of G into arbitrary finite groups. Introduce the free product $F = G * G * G$ of three copies of the group G . Let H be the free product $K * K * K$ naturally embedded in F . Evidently H vanishes under arbitrary homomorphism of the group F into a finite group.

Notice that H is a non-elementary subgroup of the hyperbolic group F , and $E_F(H) = E(F) = 1$ since there are no non-trivial finite subgroups normalized by H in the free product F . By Lemma 5.1 there exists a non-elementary hyperbolic quotient \tilde{F} of F such that $\phi(H) = \phi(F) = \tilde{F}$ for the natural epimorphism $\phi: F \rightarrow \tilde{F}$.

Assume that ψ is a homomorphism of \tilde{F} into a finite group X . Then the product $\psi\phi$ is a homomorphism of F into X . Hence $\psi\phi(H) = 1$. As far as $\phi(H) = \tilde{F}$, we get $\phi(\tilde{F}) = 1$. Consequently the group \tilde{F} has no non-trivial finite homomorphism images.

(3) \Rightarrow (2). Let G_1, \dots, G_k be a collection of non-elementary hyperbolic groups such that each common non-abelian finite simple homomorphic image of them is isomorphic to one of the groups S_1, \dots, S_n .

Introduce an auxiliary free group F_2 of rank 2, and denote by L the intersection of the kernels of all homomorphisms of F_2 into groups S_1, \dots, S_n . Notice that the set of such homomorphisms is finite. Therefore L is of finite index in the group F_2 .

Denote by H the intersection of L with the commutator subgroup $[F_2, F_2]$. Thus, H is a non-elementary subgroup of the hyperbolic group F_2 , which vanishes under any homomorphism of F_2 into any abelian group or into each of S_1, \dots, S_n . Evidently, $E_{F_2}(H) = E(F_2) = 1$.

Finally, denote by F the free product $F_2 * G_1 * \dots * G_k$. As in the part "(1) \Rightarrow (2)," Lemma 5.1 gives us a homomorphism ϕ of the group F onto a non-elementary hyperbolic group \tilde{F} , such that

$$\phi(H) = \phi(F_2) = \phi(G_1) = \dots = \phi(G_k) = \phi(F) = \tilde{F}.$$

Therefore, on the one hand, any finite simple homomorphic image $S = \psi(\tilde{F})$ is among homomorphic images of the groups G_1, \dots, G_k . So it must be either a cyclic group or one of the groups S_i for $i \leq k$. On the other hand, $S = (\psi\phi)(F_2)$, and, as was explained above, this yields the equality $(\psi\phi)(H) = 1$. Hence $S = \psi(\tilde{F}) = (\psi\phi)(H) = 1$.

The contradiction shows that the group \tilde{F} has no finite simple homomorphic images at all, and, therefore, it has no proper subgroups of finite indices.

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