Priestley Order-Compactifications

Guram Bezhanishvili and Patrick J. Morandi
Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88003-8001
gbezhani@nmsu.edu, pmorandi@nmsu.edu

Abstract

We generalize the notion of a 0-dimensional compactification of a topological space to that of a Priestley order-compactification of an ordered topological space. We also generalize the notion of a Boolean basis to that of a Priestley basis, and show that there is a 1-1 correspondence between Priestley order-compactifications and Priestley bases of an ordered topological space. This generalizes a 1961 result of Dwinger that 0-dimensional compactifications of a topological space are in a 1-1 correspondence with its Boolean bases [1].

Topic: Ordered topological spaces

Key words: Priestley order-compactification, Priestley basis
1 Introduction

We recall that a compactification of a topological space $X$ is a compact Hausdorff space $Y$ such that $X$ is homeomorphic to a dense subspace of $Y$. We also recall that a Boolean basis of $X$ is a basis of $X$ that is a field of sets. It was shown by Dwinger [1, Theorem 13.1] that the poset of 0-dimensional compactifications of $X$ is isomorphic to the poset of Boolean bases of $X$. We generalize Dwinger’s theorem to the case of ordered topological spaces. More specifically, we introduce Priestley order-compactifications and Priestley bases of an ordered topological space $X$, and show that the poset of Priestley order-compactifications of $X$ is isomorphic to the poset of Priestley bases of $X$.

2 Priestley order-compactifications

Let $X$ be an ordered topological space. We say that $X$ is order-Hausdorff if for each $x, y \in X$ with $x \not\leq y$, there exists an upset neighborhood $U$ of $x$ and a downset neighborhood $V$ of $y$ such that $U \cap V = \emptyset$ (McCartan [2]). The notion of order-Hausdorff generalizes that of a Hausdorff space to ordered topological spaces. We also say that $X$ is completely order-regular if (i) for each $x, y \in X$ with $x \not\leq y$, there exists a continuous order-preserving $f : X \rightarrow [0, 1]$ such that $f(x) > f(y)$, and (ii) for each $x \in X$ and a closed set $F$ with $x \notin F$, there exist a continuous order-preserving $f : X \rightarrow [0, 1]$ and a continuous order-reversing $g : X \rightarrow [0, 1]$ such that $f(x) = 1 = g(x)$ and $F \subseteq f^{-1}(0) \cup g^{-1}(0)$ (Nachbin [3]). The notion of completely order-regular generalizes that of a completely regular space to ordered topological spaces.

Suppose $X$ and $Y$ are ordered topological spaces. We call $Y$ an order-compactification of $X$ if $Y$ is compact order-Hausdorff and $X$ is order-homeomorphic to a dense subspace of $Y$. It follows from [3] that an ordered topological space $X$ has an order-compactification iff $X$ is completely order-regular. This generalizes a similar result about completely regular spaces and compactifications to the setting of ordered topological spaces.

Let $X$ be completely order-regular, and let $Y$ and $Z$ be two order-compactifications of $X$. We define $Y \leq Z$ if there exists a continuous order-preserving $f : Z \rightarrow Y$ such that the restriction of $f$ to $X$ is the identity function. It is easy to verify that $\leq$ is reflexive and transitive. We say that $Y$ is equivalent to $Z$ if $Y \leq Z$ and $Z \leq Y$. Then $\leq$ induces a partial order on the equivalence classes of order-compactifications of $X$.

Let $X$ be an ordered topological space. We recall that $X$ satisfies the Priestley separation axiom if for each $x, y \in X$ with $x \not\leq y$, there exists a clopen upset $U$ of $X$ such that $x \in U$ and $y \notin U$, and that $X$ is a Priestley space if $X$ is compact and satisfies the Priestley separation axiom (Priestley [4]).

Definition 2.1. Let $X$ and $Y$ be ordered topological spaces. We say that $Y$ is a Priestley order-compactification of $X$ if $Y$ is a Priestley space which is an order-compactification of $X$.
Priestley order-compactifications generalize the concept of 0-dimensional compactifications to the setting of ordered topological spaces.

For a poset $X$, let $\mathcal{U}(X)$ denote the bounded distributive lattice of upsets of $X$. We call a subset $\mathcal{R}$ of $\mathcal{U}(X)$ a ring of upsets of $X$ if $\mathcal{R}$ is a bounded sublattice of $\mathcal{U}(X)$. We say that a ring of upsets $\mathcal{R}$ is a Priestley ring if for each $x, y \in X$ with $x \nleq y$, there is $A \in \mathcal{R}$ such that $x \in A$ and $y \notin A$.

For an ordered topological space $X$, let $\mathcal{CU}(X)$ denote the ring of clopen upsets of $X$. Let also $2$ denote the ordered topological space $\{0, 1\}$ with its usual order and discrete topology. Several equivalent conditions for an ordered topological space to have a Priestley order-compactification are given in the next theorem.

**Theorem 2.2.** Let $X$ be an ordered topological space. Then the following conditions are equivalent:

1. $X$ has a Priestley order-compactification.
2. (i) $\mathcal{CU}(X)$ is a Priestley ring, and (ii) $\{U - V : U, V \in \mathcal{CU}(X)\}$ is a basis for the topology.
3. (i) $\mathcal{CU}(X)$ is a Priestley ring, and (ii) for each $x \in X$ and a closed set $F$ with $x \notin F$, there exist a clopen upset $U$ and a clopen downset $V$ with $x \notin U \cup V$ and $F \subseteq U \cup V$.
4. (i) If $x \nleq y$, then there exists a continuous order-preserving $f : X \to 2$ with $f(x) = 1$ and $f(y) = 0$, and (ii) for each $x \in X$ and a closed set $F$ with $x \notin F$, there exist a continuous order-preserving $f : X \to 2$ and a continuous order-reversing $g : X \to 2$ such that $f(x) = 1 = g(x)$ and $F \subseteq f^{-1}(0) \cup g^{-1}(0)$.

**Definition 2.3.** Let $X$ be an ordered topological space. We call a Priestley ring $\mathcal{R}$ of clopen upsets of $X$ a Priestley basis if $\{U - V : U, V \in \mathcal{R}\}$ is a basis for the topology on $X$.

Thus, $\mathcal{R}$ is a Priestley basis of $X$ if $\mathcal{R}$ is a Priestley ring that is a bounded distributive sublattice of $\mathcal{CU}(X)$, and $\{U - V : U, V \in \mathcal{R}\}$ is a basis for the topology on $X$. Let $(\mathcal{PB}(X), \subseteq)$ denote the poset of Priestley bases of $X$, and let $(\mathcal{POC}(X), \leq)$ denote the poset of inequivalent Priestley order-compactifications of $X$. Our main theorem establishes that these two posets are isomorphic.

**Theorem 2.4.** $(\mathcal{PB}(X), \subseteq)$ is isomorphic to $(\mathcal{POC}(X), \leq)$.

As a corollary we obtain Dwinger’s theorem [1]:

**Theorem 2.5 (Dwinger).** The poset $(\mathcal{Z}(X), \leq)$ of inequivalent 0-dimensional compactifications of $X$ is isomorphic to the poset $(\mathcal{B}(X), \subseteq)$ of Boolean bases of $X$. 


References


