Abstract. MV-algebras of various kinds have been heavily investigated in the recent times. The isomorphism theorems between the MV-algebras and the interval MV-algebras in the corresponding lattice-ordered algebraic structures support research that utilizes the established properties of these structures in order to obtain specific information about the initial MV-algebras. In this talk we will discuss a concrete shape of any product MV-algebra that naturally embeds in a real algebra of matrices. We will use the structure theorem about lattice-ordered real algebras of matrices. In particular we establish that PMV-algebras in concern are precisely the intervals between the zero matrix and a conjugate of a certain positive matrix.

Keywords: PMV-algebra, MV-module, lattice-ordered algebra of matrices, strong order unit, full cone
Theorem 0.1. Let \((M, \oplus, \cdot, 0, 1)\) be a product MV-algebra with \(M \subseteq \mathbb{R}_n\), i.e. \(M\) is a collection of \(n \times n\) matrices over \(\mathbb{R}\). Assume additionally that:

i. \(M\) is a standard MV-module over \([0, 1]\);

ii. the partial addition \(A + B\), the scalar multiplication by elements form \([0, 1]\) and the product \(A \cdot B\) of elements of \(M\) are standard matrix operations;

iii. \(M\) generates \(\mathbb{R}_n\) in the sense that every element of \(\mathbb{R}_n\) is a real multiple of a difference of two disjoint elements from \(M\) (i.e. if \(A \in \mathbb{R}_n\), then there is \(\alpha \in \mathbb{R}\) and \(K, L \in M\) such that \(K \land L = 0\) and \(\alpha A = K - L\); here “\(\land\)” is the MV-algebra infimum operation.)

Then there exists a nonsingular nonnegative (in the usual sense) matrix \(H\), a nonsingular matrix \(C\) and a matrix \(W = \sum_{i,j=1}^n \omega_{ij} C^{-1} E_{ij} H^T C\) with all \(\omega_{ij} > 0\), such that every matrix \(A \in M\) has the form

\[ A = \sum_{i,j=1}^n \alpha_{ij} C^{-1} E_{ij} H^T C \]

with

\[ 0 \leq \alpha_{ij} \leq \omega_{ij} \]

for \(i, j = 1, \ldots, n\). (Here \(E_{ij}\) denotes the matrix with 1 in the \(ij\) entry and zeros elsewhere.)

Conversely, if \(H, W\) and \(C\) are as above then there exists a number \(\mu > 0\) such that \(\Gamma((\mathbb{R}_n, C^{-1} P_H C), \mu W)\) is a product MV-algebra.

Throughout we use the notation of \((\mathbb{R}_n, C^{-1} P_H C)\) to indicate the lattice-ordered real algebra \(\mathbb{R}_n\) with the positive cone equal precisely \(\sum_{i,j=1}^n \mathbb{R}^+ C^{-1} E_{ij} H^T C\). It is proven in Ma and Wojciechowski [4] that any lattice-ordered algebra \(\mathbb{R}_n\) is of this form. Particular structure of the basic elements of those orders are discussed there.

We need the following lemma which is applicable to MV-modules in general.

Lemma 0.2. Let \(M\) be a standard \([0, 1]\)-MV-module. Then for any two \(a, b \in M\) there exists \(\alpha > 0\) such that the partial addition \(\alpha a + \alpha b\) is defined.

Proof. By Di Nola et all. [3] Corollary 4.2 there exists a standard \([0, 1]\) l-module \(G\) possessing a strong order unit \(u\), such that the MV-modules \(M\) and \(\Gamma(G, u)\) are isomorphic. If we ignore the notational differences between the corresponding elements of the two MV-modules, we can say that since \(u\) is a strong order unit in \(G\), there is \(n\) such that \(a + b \leq nu\). By letting \(\alpha = \frac{1}{n}\) we have \(\alpha a \leq u - \alpha b\), which is equivalent to saying \(\alpha a \leq (ab)^*\). The same happens in \(M\), and therefore the partial addition \(\alpha a + \alpha b\) is defined in \(M\). \(\square\)

Lemma 0.3. Let \(V\) be a real finite-dimensional vector space and let \(P\) be a full cone in \(V\) (i.e. an algebraically and topologically closed and generating cone in \(V\)). With respect to the partial order determined by the cone, an element \(v \in V\) is a strong order unit if and only if \(v \in \text{int}(P)\).

Proof. (sketch) We pick up \(v \in \text{int}(P)\) and an arbitrary element \(u \in P\). We next reduce the arguments to a full two dimensional cone and by geometric arguments we show that \(u \leq \alpha v\) for a suitable nonnegative \(\alpha\). Similar arguments prove the converse. \(\square\)
Theorem 0.4. In a lattice-ordered algebra of matrices, $\mathbb{R}_n$, all strong order units $W$ are of the form $W = \sum_{i,j=1}^{n} \omega_{ij} C^{-1} E_{ij} H^T C$ for $H$ nonsingular and nonnegative (in the usual sense), $C$ nonsingular and all $\omega_{ij} > 0$. Moreover, there exists a positive constant $\mu$ such that $\mu W$ is a strong order unit satisfying the condition $(\mu W)^2 \leq \mu W$.

Proof. (sketch) This is a corollary from Lemma 0.3 and from the fact that the positive cone of any lattice order of $\mathbb{R}_n$ is a (simplicial) full cone $\sum_{i,j=1}^{n} \mathbb{R}^+ C^{-1} E_{ij} H^T C$ ([4]). In the second part, the constant $\mu$ can be easily calculated based on the dimension $n$ and the ranges of the entries of $H$ and $W$. \hfill \Box

As an immediate corollary we obtain a useful statement about the strong order units.

Theorem 0.5. A matrix $W$ is a strong order unit in a lattice-ordered algebra $\mathbb{R}_n$ if and only if $W$ is similar to a positive matrix (in the usual sense). For such $W$ there exists a positive constant $\mu$ such that $(\mu W)^2 \leq \mu W$.

Proof. (sketch, the main theorem)

Let $P = \{ \alpha A : \alpha \geq 0$ and $A \in M \}$. We show that $P$ is a positive cone of a lattice-ordered real algebra $\mathbb{R}_n$. Let $\alpha A, \beta B \in P$ and let $\gamma > 0$ be a real number such that $\gamma A + \gamma B$ is defined in $M$ (existence of such $\gamma$ follows from Lemma 0.2.) Then $\alpha A + \beta B = \gamma (\alpha A + \beta B) \in P$. Moreover, the condition $P \cap -P = \{0\}$ is satisfied immediately because two elements from $M$ add to 0 via the partial addition if and only if both are zero. Since also $\alpha A \cdot \beta B = \gamma^2 (\alpha A \cdot \beta B) \in P$, and $\alpha (\beta B) = (\alpha \beta) B \in P$, $P$ is a positive cone of a partially ordered real algebra $\mathbb{R}_n$. It is a lattice-ordered algebra because of the condition (iii) and Proposition 4.3 of Darnel [1].

This way $M$ becomes a sublattice of a lattice-ordered algebra of matrices $\mathbb{R}_n$. It follows from [4] that $P = C^{-1} P H C$ for some nonsingular nonnegative matrix $H$ and a nonsingular matrix $C$. We now apply the fundamental theorem 3.2 of Di Nola and Dvurecenskij from [2] to argue that with respect to an appropriate lattice order of this kind, there exists a strong order unit matrix $W$ with $W^2 \leq W$ so that $M = [0, W]$. Therefore $W$ is a matrix described in Theorem 0.4. Thus $W = \sum_{i,j=1}^{n} \omega_{ij} C^{-1} E_{ij} H^T C$ with all $\omega_{ij} > 0$, and the rest of theorem follows.

The converse follows from [2] Theorem 3.2 and our Theorem 0.4. \hfill \Box

By Theorem 0.5 the second part of the above theorem can be expressed as:

Theorem 0.6. $\Gamma(\mathbb{R}_n, \mu W)$ is a product MV-algebra if and only if $W$ is similar to a positive matrix.

It is understood that $\mathbb{R}_n$ is given some lattice order and the constant $\mu$ is appropriately chosen.

References