Relevance logic and the calculus of relations

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Summary

Early papers on relevance logic are Church 1951 [5], Moh 1950 [11], and Ackermann 1956 [1]. Four primary collections of work on relevance logics are Anderson-Belnap 1975 [2], Anderson-Belnap-Dunn 1992 [3], Routley-Plumwood-Meyer-Brady 1982 [16], and Brady 2003 [4]. Semantics for relevance logics were developed by Fine 1974 [8], Routley-Meyer 1973 [15], and Urquhart 1972 [17] and were very extensively elaborated, varied, and applied to a multitude of systems over the last three decades; see Routley-Routley 1972 [12], Routley-Meyer 1972 [14], Routley-Meyer 1972 [13], Meyer-Routley 1973–74 [10], and especially Routley-Plumwood-Meyer-Brady 1982 [16]. The relevant model structures introduced by Routley-Meyer [15] are closely related to the atom structures of relation algebras, a connection also noted by Dunn 2001 [6] and Dunn-Restall 2001 [7]. By combining this connection with the notion of representability for relation algebras we arrive at new semantics in which formulas are interpreted as binary relations, similar to the manner in which formulas may be interpreted as sets to obtain sound and complete semantics for the classical propositional calculus. This paper contains some elementary remarks and questions about the interpretation of formulas as binary relations and about the connections betweeen relevant model structures and atom structures of relation algebras.

Two of the primary systems of relevance logic are **R** and **RM**. Formulas are built up from variables using connectives $\to$, $\sim$, $\wedge$, $\vee$. To get an axiom set for **RM**, add $A \to (A \to A)$ to an axiom set for **R**. $A \to (A \to A)$ is called *Mingle*. The rules of deduction for **R** and **RM** are *adjunction* $A,\ B \vdash A \wedge B$ and *modus ponens* $A,\ A \to B \vdash B$. Given a set U and binary relations $A, B, C \subseteq U^2$, define

$$A \wedge B := A \cap B \quad A \vee B := A \cup B \tag{1}$$

$$\sim A := \{\langle x, y\rangle : x, y \in U \text{ and } \langle y, x\rangle \notin A\} \tag{2}$$

$$A \to B := \{\langle x, z\rangle : \forall_y(\text{ if } yAx \text{ then } yBz)\} \tag{3}$$

$$\mathsf{Id} := \{\langle x, x\rangle : x \in U\} \tag{4}$$

$$A|B := \{\langle x, z\rangle : \exists_y(\langle x, y\rangle \in A \text{ and } \langle y, z\rangle \in B)\} \tag{5}$$

Let $\mathcal{R} \subseteq Sb(U^2)$. Some possible additional assumptions on $\mathcal{R}$ are

(comm) $A|B = B|A$ for all $A, B \in \mathcal{R}$,
(dense) $A \subseteq A|A$ for every $A \in \mathcal{R}$,
(trans) $A|A \subseteq A$ for every $A \in \mathcal{R}$.

A formula A is *true in* $\mathcal{R}$ if $\mathsf{Id} \subseteq A$ whenever the variables of A are interpreted as relations in $\mathcal{R}$ and the connectives of A are interpreted as operations on sets according to the definitions (1), (2), (3). Let

$$K_{\mathbf{R}} := \{\mathcal{R} : \exists_U(\mathsf{Id} \in \mathcal{R} \subseteq \mathfrak{Re}\,(U)\,, (\text{comm}), (\text{dense}))\}$$
$$K_{\mathbf{RM}} := \{\mathcal{R} : \exists_U(\mathsf{Id} \in \mathcal{R} \subseteq \mathfrak{Re}\,(U)\,, (\text{comm}), (\text{dense}), (\text{trans}))\}$$

Theorem 1

(i) If A is a theorem of **R** then A is true in every $\mathcal{R} \in K_{\mathbf{R}}$.
(ii) If A is a theorem of **RM** then A is true in every $\mathcal{R} \in K_{\mathbf{RM}}$.

It is easy to construct a set of relations such that (comm) and (dense) but not (trans) (see the example below), so Mingle cannot be proved in **R**.

Prob. If A is not a theorem of **R** (or of **RM**), must A fail to be true in some $\mathcal{R} \in K_{\mathbf{R}}$ (in some $\mathcal{R} \in K_{\mathbf{RM}}$)?

The proof of Th. 1 can be based directly on the set-theoretical definitions, or can be done by pure calculation, using formulas from the calculus of relations which happen to hold in all relation algebras. Consequently the (appropriately restated) Th. 1 still holds if formulas are interpreted as elements of relation algebras. The existence of nonrepresentable relation algebras suggests that completeness may fail for formulas-as-relations. All that is required to prove this for **R** is to find a relation-algebraic equation E with these properties:

(i) the only operations occurring in E are the relation-algebraic operations which correspond to $\to$, $\sim$, $\vee$, and $\wedge$,
(ii) E holds in every representable dense commutative relation algebra,
(iii) E fails in some commutative dense relation algebra.

The currently known equations with the last two properties do not have the first. Such an equation would yield a formula A which is true in every $\mathcal{R} \in K_{\mathbf{R}}$ and yet is not a theorem of **R**.

Example. Let U be the set of rational numbers, let $L = \{\langle x, y\rangle : x < y\}$ and let $G = \{\langle x, y\rangle : x > y\}$. The relation algebra of all binary relations on U has a subalegbra generated by L and G. It contains the relations $\mathcal{R} = \{\emptyset, \mathsf{Id}, \mathsf{Id} \cup L, U^2, \mathsf{Id} \cup G, L, L \cup G, G\}$. All the relations in $\mathcal{R}$ are dense and commute with each other, so $\mathcal{R} \in K_{\mathbf{R}}$, but $L \cup G$ is not transitive, so $\mathcal{R} \notin K_{\mathbf{RM}}$. There are nine nonempty subsets of $\mathcal{R}$ which are closed under $\to$, $\sim$, $\wedge$, and $\vee$. Two of them are $\mathcal{R}_0 := \{L, \mathsf{Id} \cup L\}$ and $\mathcal{R}_1 := \{G, \mathsf{Id} \cup G\}$. Note that $X \to Y = \emptyset$ for all $X \in \mathcal{R}_0$

and $Y \in \mathcal{R}_1$. Also, if the variables of a formula A are interpreted as elements of $\mathcal{R}_0$ then A itself will be interpreted as an element of $\mathcal{R}_0$ since $\mathcal{R}_0$ is closed under $\vee$, $\wedge$, $\rightarrow$, and $\sim$ (and similarly for $\mathcal{R}_1$). Suppose the formulas A and B share no variable. Interpret the variables of A as any elements of $\mathcal{R}_0$, and interpret the variables of B as any elements of $\mathcal{R}_1$. Then $A \in \mathcal{R}_0$ and $B \in \mathcal{R}_1$, so $A \rightarrow B = \emptyset$, hence $A \rightarrow B$ is not true in $\mathcal{R}$. But $\mathcal{R} \in K_{\mathbf{R}}$ so by Th. 1 $A \rightarrow B$ is not a theorem of $\mathbf{R}$. This proof (that if $A \rightarrow B$ is a theorem of $\mathbf{R}$ then A and B share a variable) occurs at the end of Routley-Meyer 1973 [15]. Instead of Id, L, and G, they use elements in a particular relevant model structure $\mathcal{M}$. But $\mathcal{M}$ happens to be the atom structure of the representable relation algebra 1_3 (see [9, §56.6]) and $\mathcal{R}$ is simply the set of relations used in a particular representation of 1_3.

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