Non-polynomial Polar Forms

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Abstract. We begin by defining the polar form for a special type of function, namely a trigonometric polynomial, in order to illustrate the similarities between trigonometric polar forms and polynomial polar forms. After deriving properties and developing some results concerning trigonometric polar forms, we consider the generalization to functions that are elements of certain null spaces of constant coefficient differential operators.

§1. Introduction

The concept of a polar form or blossom, while known for quite some time in an algebraic context, has been introduced into the spline theory by de Casteljau and independently by Ramshaw (see [6], for a comprehensive introduction). Polar forms have proven to be a convenient mathematical tool for describing (piecewise) polynomial functions and for analyzing various spline algorithms such as recurrence relations and knot insertion [3,6,8].

A generalization of polar forms to non-polynomial functions has been given in [5]. There, a geometric approach to polar forms has been developed, whereas our generalization is based on the fact that polynomials form a translation invariant space. Since translation invariant spaces are the null spaces of constant coefficient differential operators, it follows that the polar forms defined here are multivariate exponential polynomials. Hence, these polar forms are *not* multi-affine functions in the classical sense. Nevertheless, they have a similar structure to polynomial polar forms and therefore could be useful in a context of a general non-polynomial spline theory.

We begin our presentation by considering polar forms for trigonometric functions in Section 2. This will motivate our approach in the more general situation described in Section 3.

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§2. Trigonometric polar forms

In this section we will define trigonometric polar forms and discuss some of their properties. First we recall some basic definitions and notations. We will let $\mathcal{T}_n$ denote the following space of *trigonometric polynomials of order* $n+1$, $(n \geq 0)$:

$$\mathcal{T}_n := \begin{cases} \text{span}\{1, \lfloor 2x \rfloor, \lceil 2x \rceil, \lfloor 4x \rfloor, \lceil 4x \rceil, \ldots, \lfloor nx \rfloor, \lceil nx \rceil\}, & n \text{ even} \\ \text{span}\{\lfloor x \rfloor, \lceil x \rceil, \lfloor 3x \rfloor, \lceil 3x \rceil, \ldots, \lfloor nx \rfloor, \lceil nx \rceil\}, & n \text{ odd,} \end{cases}$$

where, for the sake of shortening the notation, we defined

$$\lfloor x \rfloor := \sin x, \quad \lceil x \rceil := \cos x.$$

As is well known ([7]), the space $\mathcal{T}_n$ can be identified with the kernel of the differential operator D_n, defined by

$$D_n := \begin{cases} D(D^2+2^2)(D^2+4^2)\cdots(D^2+n^2), & n \text{ even} \\ (D^2+1^2)(D^2+3^2)\cdots(D^2+n^2), & n \text{ odd,} \end{cases}$$

where $D := d/dx$. Another equivalent way of defining the space $\mathcal{T}_n$ is

$$\mathcal{T}_n = \text{span}\left\{ \binom{n}{k} \lfloor x \rfloor^{n-k} \lceil x \rceil^k \right\}_{k=0}^{n},$$

which justifies the term *degree* used in the context of trigonometric polynomials for the number n. It should be remarked that in the definition of the space $\mathcal{T}_n$ we have deviated slightly from the convention used in the literature on trigonometric splines. It is common to define spaces of trigonometric polynomials using trigonometric functions with halved arguments. For example, the space $\mathcal{T}_2$ is most often taken to be the space, $\text{span}\{1, \sin x, \cos x\}$ rather than the above defined, $\text{span}\{1, \sin 2x, \cos 2x\}$. However, the convention followed in this paper reflects the results in [1], suggesting that the definition of the space $\mathcal{T}_n$ given here is more natural.

We are ready to introduce the polar form of a trigonometric function.

Theorem 1. *For every $F \in \mathcal{T}_n, n \geq 0$, there exists a unique function $f(x_1, \ldots, x_n)$ of n variables, $x_1, \ldots, x_n \in \mathbb{R}$, called the trigonometric polar form of F, which satisfies the following properties:*

(a) f is symmetric with respect to $x_1, \ldots, x_n$,
(b) f is equal to F on the diagonal *i.e.*, $f(x, \ldots, x) = F(x)$, for all $x \in \mathbb{R}$,
(c) for all $m \geq 1$ and all real numbers $y, y_1, \ldots, y_m$, the function f satisfies in each variable the relation

$$f(\ldots, y, \ldots) = \sum_{j=1}^{m} \lambda_j f(\ldots, y_i, \ldots), \tag{7}$$

whenever the numbers $\lambda_1, \dots, \lambda_m$ are chosen so that

$$\sum_{j=1}^{m} \lambda_j \lfloor y_j \rfloor = \lfloor y \rfloor, \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j \lceil y_j \rceil = \lceil y \rceil. \tag{8}$$

Proof: Let $\alpha := (\alpha_1, \dots, \alpha_n)$ be a multi-index, where $\alpha_i \in \{0,1\}, i = 1, \dots, n$, and $1 - \alpha := (1-\alpha_1, \dots, 1-\alpha_n)$. Moreover, for $X := (x_1, \dots, x_n)$, we denote $\lfloor X \rfloor^{\alpha} := \lfloor x_1 \rfloor^{\alpha_1} \cdots \lfloor x_n \rfloor^{\alpha_n}$. First observe that the function

$$s_k(x_1, \dots, x_n) := \sum_{|\alpha|=k} \lfloor X \rfloor^{1-\alpha} \lceil X \rceil^{\alpha},$$

is a polar form of the function $\binom{n}{k} \lfloor x \rfloor^{n-k} \lceil x \rceil^{k} \in \mathcal{T}_n$. This is easily proved by verifying all three defining properties (a)–(c). While the proofs of the first two properties are straightforward, the third property follows from the fact that the functions $\lfloor X \rfloor^{1-\alpha} \lceil X \rceil^{\alpha}$ satisfy relation (7). This becomes clear when one observes that these functions are each a product of one of the univariate functions $\lfloor x_i \rfloor$ or $\lceil x_i \rceil$, and both of these functions trivially satisfy (7) under assumption (8).

Next, since the functions $\binom{n}{k} \lfloor x \rfloor^{n-k} \lceil x \rceil^{k}, k = 0, \dots, n$, form a basis for $\mathcal{T}_n$, we conclude that the function

$$f(x_1, \dots, x_n) = \sum_{k=0}^{n} c_i s_k(x_1, \dots, x_n), \quad c_0, \dots, c_n \in \mathbb{R}, \tag{9}$$

is a polar form of $F \in \mathcal{T}_n$, given by

$$F(x) = \sum_{k=0}^{n} c_i s_k(x, \dots, x) = \sum_{k=0}^{n} c_i \lfloor x \rfloor^{n-k} \lceil x \rceil^{k}.$$

For the uniqueness of representation (9), it suffices to notice that the n-variate functions $s_k(x_1, \dots, x_n), k = 0, \dots, n$, are linearly independent since they are linearly independent on their diagonal $x_1 = \dots = x_n$. ■

Remark 1. *On account of the symmetry of f, the above recursion (7) could be called a trigonometric multi-affineness of f.*

Remark 2. *It is possible to relate polynomial polar forms to trigonometric polar forms. Let $F \in \mathcal{T}_n, n \geq 0$, and let P be such that*

$$P\left(\frac{\lfloor x \rfloor}{\lceil x \rceil}\right) = \frac{F(x)}{\lceil x \rceil^{n}}, \quad \lceil x \rceil \neq 0.$$

Then $P \in \Pi_n$, Π_n the space of (algebraic) polynomials of order $n+1$, and so it has a polar form, say p. The function

$$f(x_1, \dots, x_n) := \lceil x_1 \rceil \dots \lceil x_n \rceil p\left(\frac{\lfloor x_1 \rfloor}{\lceil x_1 \rceil}, \dots, \frac{\lfloor x_n \rfloor}{\lceil x_n \rceil}\right)$$

coincides with the trigonometric polar form of F, for all $x_1, \ldots, x_n$, *such that* $\lceil x_1 \rceil \ldots \lceil x_n \rceil \neq 0$.

Remark 3. *Let* $\sigma_k(x_1, \ldots, x_n)$ *be the* k*-th elementary symmetric polynomial of* n *arguments. Recall that* $\sigma_k(x_1, \ldots, x_n)$ *is the (polynomial) polar form of the monomial* $\binom{n}{k}x^k$. *Therefore, by Remark 2, we have*

$$s_k(x_1, \ldots, x_n) = \lceil x_1 \rceil \cdots \lceil x_n \rceil \sigma_k \left(\frac{\lfloor x_1 \rfloor}{\lceil x_1 \rceil}, \ldots, \frac{\lfloor x_n \rfloor}{\lceil x_n \rceil} \right),$$

and hence $s_k(x_1, \ldots, x_n)$ *can be viewed as an elementary symmetric trigonometric polynomial. It follows from the proof of Theorem 1 that the trigonometric polar form of any trigonometric polynomial is a linear combination of the elementary symmetric trigonometric polynomials.*

As a consequence of Theorem 1, we have the next three-term recurrence relation for polar forms.

Corollary 1. *Let* f *be the trigonometric polar form of* $F \in \mathcal{T}_n$. *Then*

$$f(\ldots, y, \ldots) = \frac{\lfloor y_2 - y \rfloor}{\lfloor y_2 - y_1 \rfloor} f(\ldots, y_1, \ldots) + \frac{\lfloor y - y_1 \rfloor}{\lfloor y_2 - y_1 \rfloor} f(\ldots, y_2, \ldots), \tag{10}$$

for all $y, y_1, y_2 \in \mathbb{R}$, *such that* $\lfloor y_2 - y_1 \rfloor \neq 0$.

Proof: Setting $m = 2$, the two equations in (8) represent a linear system for the unknowns λ_1 and λ_2, which by elementary algebra leads to

$$\lambda_1 = \frac{\lfloor y_2 - y \rfloor}{\lfloor y_2 - y_1 \rfloor}, \quad \lambda_2 = \frac{\lfloor y - y_1 \rfloor}{\lfloor y_2 - y_1 \rfloor}. \quad \blacksquare$$

Remark 4. *Relation (10) could be considered as an alternative to (7). Indeed, it is not difficult to show that if a function* f *satisfies (10) for all admissible choices of* y, y_1, y_2, *then it also satisfies the more general recursion (7).*

Next, we present an analog of a property of polynomial polar forms [6].

Theorem 2. *(Polar Interpolation) Let* $a_0, a_1 \in \mathbb{R}$, $0 < |a_1 - a_0| < \pi$. *For* $k = 0, \ldots, n$, *let* $c_k \in \mathbb{R}$ *and* $t_k := (\overbrace{a_0, \ldots, a_0}^{n-k}, \overbrace{a_1, \ldots, a_1}^{k})$. *Then there exists a unique trigonometric polynomial* F *of degree* n *whose polar form* f *satisfies* $f(t_k) = c_k, k = 0, \ldots, n$.

Proof: Let us first assume that $|a_1 - a_0| \neq \pi/2$. We define the functions b_0 and b_1 by

$$b_0(x) := \frac{\lfloor a_1 - x \rfloor}{\lfloor a_1 - a_0 \rfloor} \quad \text{and} \quad b_1(x) := \frac{\lfloor x - a_0 \rfloor}{\lfloor a_1 - a_0 \rfloor}. \tag{11}$$

Clearly,

$$b_j(a_k) = \delta_{jk}, \quad j,k = 0,1. \tag{12}$$

Next, let

$$F(x) := \sum_{k=0}^{n} c_k B_k^n(x), \tag{13}$$

where $B_k^n(x) := \binom{n}{k} b_0^{n-k}(x) b_1^k(x)$. It is not difficult to prove that the functions, $B_k^n, k = 0, \ldots, n$, are linearly independent. This can be done *e.g.*, by induction on n. It is clear from (12) that the assertion is true for $n = 1$ since in this case, $t_k = a_k, k = 0,1$. For $n > 1$, let $\sum_{k=0}^{n} d_i B_k^n(x) = 0$, for all $x \in \mathbb{R}$. In particular, this equality must hold for $x = a_0$. Therefore, by (12), $d_0 = 0$. However, the remaining sum is now a product of the function b_1 with a linear combination of functions $B_k^{n-1}, k = 0, \ldots, n-1$, which are linearly independent by the induction hypothesis. Therefore, the remaining coefficients d_k, for $k = 1, \ldots, n$ must be zero. Hence, the B_k^n are linearly independent and thus the representation of F in (13) is indeed unique.

Next, let f be the polar form of F. We show that the function f satisfies the interpolation conditions $f(t_k) = c_k$. Observe that for an arbitrary but fixed number x, the value $F(x) = f(x, \ldots, x)$ can be expressed as a linear combination of the values $f(t_k)$. For example, applying (10) to the first argument of f leads to

$$f(x, \ldots, x) = b_0(x) f(a_0, x, \ldots, x) + b_1(x) f(a_1, x, \ldots, x).$$

In general, we obtain

$$F(x) = f(x, \ldots, x) = \sum_{k=0}^{n} f(t_k) B_k^n(x), \tag{14}$$

which, in combination with (13), gives the desired result. The uniqueness of the function F satisfying the interpolation conditions follows from representation (14), since $c_k = 0, k = 0, \ldots, n$, clearly forces F to be the zero function.

The proof for the remaining case $|a_1 - a_0| = \pi/2$ is almost identical with the above proof except that now the definitions of the functions b_0 and b_1 have to be modified. Assuming, without loss of generality that $a_1 > a_0$, we can set

$$b_0(x) := \lfloor a_1 - x \rfloor \quad \text{and} \quad b_1(x) := \lfloor x - a_0 \rfloor = \lceil a_1 - x \rceil. \quad \blacksquare \tag{15}$$

Corollary 2. *The functionals $\mu_j : \mathcal{T}_n \to \mathbb{R}$, $j = 0, \ldots, n$, defined by*

$$\mu_j F := f(t_j), \quad F \in \mathcal{T}_n, \quad f \text{ – the polar form of } F,$$

form a dual basis for $\{B_k^n\}_{k=0}^n$ i.e.,

$$\mu_j B_k^n = \delta_{jk}, \quad j,k = 0, \ldots, n.$$

Remark 5. *The above results suggest that the trigonometric functions* B_k^n *associated with the interval* $[a_0, a_1]$ *may be viewed as analogs of classical Bernstein polynomials. These trigonometric Bernstein polynomials have been studied in greater detail in [1]. In that paper they have been coined circular Bernstein polynomials since the functions* b_0 *and* b_1 *defined by (11) and (15) can be considered as circular analogs of barycentric coordinates. From Corollary 2 it follows that, as in the polynomial case, the coefficients* $c_k, k = 0, \ldots, n$ *of a trigonometric polynomial of the form (13) can be obtained by evaluating the trigonometric polar form* f *of* F *at the points* t_k.

§3. Polar forms for certain translation invariant spaces

In this section we briefly describe how the results of Section 2 can be carried over to a larger class of functions. Let us first observe that the equations (8) are equivalent to

$$\sum_{j=1}^{m} \lambda_j \lfloor y_j - t \rfloor = \lfloor y - t \rfloor, \tag{16}$$

which must hold true for all $t \in \mathbb{R}$. This equation suggests the following generalization of (8). Let d be a real-valued function. In accordance with (16), we require that every $y \in \mathbb{R}$ can be associated with numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\sum_{j=1}^{m} \lambda_j d(y_j - t) = d(y - t), \tag{17}$$

for all $t \in \mathbb{R}$, provided equation (17) is solvable. In particular, we require that it be solvable for $m = 2$, whenever the two functions $d(y_j - \cdot), j = 1, 2$ are linearly independent. This requirement imposes strong restrictions on the function d. To explain this, it will be convenient to introduce the space $\mathcal{D} := \operatorname{span}\{d(\cdot - t), t \in \mathbb{R}\}$, *i.e.*, the linear span of all translates of the function d. Setting $m = 2$ and keeping y, y_1, y_2 fixed, equation (17) implies that every function from $\mathcal{D}$ can be expressed as a linear combination of two fixed functions. Hence, the dimension of $\mathcal{D}$ can be at most two. Another restriction on $\mathcal{D}$ is that it must be a translation invariant space, that is, such that $d \in \mathcal{D}$ implies $d(\cdot - t) \in \mathcal{D}$, for all $t \in \mathbb{R}$. The translation invariance is clearly a consequence of the definition of $\mathcal{D}$. The case, where the dimension of $\mathcal{D}$ is one, is trivial since the only translation invariant one dimensional space is the space of constant functions. In the remainder of this section we will only consider the case where the dimension of $\mathcal{D}$ is two. The space $\mathcal{D}$ is completely characterized by the following

Proposition 1. *A two dimensional space of continuous real-valued functions is translation invariant if and only if it is the null space of a linear second order constant coefficient differential operator.*

Proof: The crux of the proof is in treating the elements of the space $\mathcal{D}$ under consideration as distributions. It is easily seen that a two dimensional translation invariant space $\mathcal{D}$ of distributions is also invariant under differentiation *i.e.*, $f \in \mathcal{D}$ implies $f' \in \mathcal{D}$. Next, let $f \in \mathcal{D}$ such that f and f' are linearly independent. The existence of such an f can be proved as follows. Suppose, on the contrary, that there is no such function. Let f_1, f_2 be two linearly independent elements of $\mathcal{D}$ and let a_1, a_2 be real numbers such that $f_1' = a_1 f_1, f_2' = a_2 f_2$. Moreover, let $f \in \mathcal{D}$ and let a be such that $f' = af$. Thus, $f = b_1 f_1 + b_2 f_2$ for some real coefficients b_1, b_2. By combining these equalities and by the linear independence of f_1, f_2 it follows that $a = a_1 = a_2$. Therefore, since f was arbitrary, $\mathcal{D}$ is the solution space of the equation $f' - af = 0$, which is a one dimensional space. This contradicts the assumption that $\mathcal{D}$ is two dimensional. Therefore, let f be such that f and f' are independent *i.e.*, such that they span $\mathcal{D}$. Since f'' is an element of $\mathcal{D}$, there exist two coefficients a, b such that $f'' = af' + bf$. Thus, f solves a constant coefficient differential homogeneous equation. However, then f' must also solve the same equation which can be easily seen by differentiating both sides of this equation. Since f and f' are independent it follows that all elements of $\mathcal{D}$ must solve this equation. To finish the proof, it is sufficient to realize that the null space of a second order constant coefficient differential operator is translation invariant. ■

Remark 6. *The assertion of Proposition 1 also follows from [2, Thm. 1.3 (a)]. In fact, the assumption of continuity of the functions in $\mathcal{D}$ is unnecessarily strong. It is sufficient to assume that $\mathcal{D}$ is a space of distributions.*

In the sequel, let $\mathcal{D}$ be the null space of a second order constant coefficient differential operator and let $\mathcal{D}_n := \text{span}\{d^n, d \in \mathcal{D}\}$. In particular, $\mathcal{D}_0$ is the space of constant functions. We are ready to define a $\mathcal{D}$-polar form of a function $F \in \mathcal{D}_n$.

Theorem 3. *For every $F \in \mathcal{D}_n$, $n \geq 0$ there exists a unique function $f(x_1, \ldots, x_n)$ of n variables, called a $\mathcal{D}$-polar form of F, satisfying the following properties:*

(a) f is symmetric with respect to $x_1, \ldots, x_n$,
(b) f is equal to F on the diagonal *i.e.*, $F(x) = f(x, \ldots, x)$, for all $x \in \mathbb{R}$,
(c) for all $m \geq 1$ and all real numbers $y, y_1, \ldots, y_m$, the function f is $\mathcal{D}$-*affine* *i.e.*, it satisfies in each variable the relation

$$f(\ldots, y, \ldots) = \sum_{j=1}^{m} \lambda_j f(\ldots, y_i, \ldots), \tag{20}$$

whenever the coefficients $\lambda_1, \ldots, \lambda_m$ are chosen such that

$$\sum_{j=1}^{m} \lambda_j d(y_j) = d(y), \tag{8}$$

for all $d \in \mathcal{D}$.

Proof: The proof can be done along the same lines as the proof of Theorem 1. Here, the functions $\lfloor x \rfloor$ and $\lceil x \rceil$ should be replaced by two arbitrary linearly independent functions from $\mathcal{D}$. ■

Remark 7. *The space $\mathcal{D}_n$ arises in a different context also in [4]. There, spline spaces are considered whose elements can be locally identified with functions from $\mathcal{D}_n$. The associated B-splines turn out to satisfy recurrence relations which are similar to the classical ones for polynomial B-splines.*

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