

Noncommutative geometry, the
field with one element, and
3-manifolds

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The field with one element

Finite geometries ($q = p^k$, p prime)

$$\#\mathbb{P}^{n-1}(\mathbb{F}_q) = \frac{\#(\mathbb{A}^n(\mathbb{F}_q) \setminus \{0\})}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q$$

$$\#\mathrm{Gr}(n, j)(\mathbb{F}_q) = \#\{\mathbb{P}^j(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q)\}$$

$$= \frac{[n]_q!}{[j]_q![n-j]_q!} = \binom{n}{j}_q$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad [0]_q! = 1$$

J. Tits: take $q = 1$

$\mathbb{P}^{n-1}(\mathbb{F}_1) :=$ finite set of cardinality n

$\mathrm{Gr}(n, j)(\mathbb{F}_1) :=$ set of subsets of cardinality j

Algebraic geometry over \mathbb{F}_1 ?

Extensions \mathbb{F}_1^n (Kapranov-Smirnov)

Monoid $\{0\} \cup \mu_n$ (n-th roots of unity)

- Vector space over \mathbb{F}_1^n : pointed set (V, v) with free action of μ_n on $V \setminus \{v\}$
- Linear maps: permutations compatible with the action

$$\mathbb{F}_1^n \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}] / (t^n - 1)$$

Zeta functions (Manin) as variety over \mathbb{F}_1 :

$$Z(\mathbb{T}^n, s) = \frac{s - n}{2\pi} \quad \text{with } \mathbb{T}^n = \mathbb{G}_m^n$$

$\Rightarrow \mathbb{F}_1$ -geometry and infinite primes (Arakelov)

$$\Gamma_{\mathbb{C}}(s)^{-1} = ((2\pi)^{-s} \Gamma(s))^{-1} = \prod_{n \geq 0} \frac{s + n}{2\pi}$$

like a “dual” infinite projective space $\bigoplus_{n=0}^{\infty} \mathbb{T}^{-n}$ over \mathbb{F}_1

Note: $\text{Spec}(\mathbb{Z})$ not a (fin. type) scheme over \mathbb{F}_1 : what is $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$??

More recently: Soulé, Haran, Dourov, Toën–Vaquie, Connes–Consani, Borger

Focus on two approaches:

Descent data for rings from \mathbb{Z} to \mathbb{F}_1 :

- by cyclotomic points (Soulé)
- by Λ -ring structure (Borger)

Gadgets over \mathbb{F}_1

C. Soulé, *Les variétés sur le corps à un élément*, Mosc. Math. J. Vol.4 (2004) N.1, 217–244.

$$(X, \mathcal{A}_X, e_{x,\sigma})$$

- $X : \mathcal{R} \rightarrow \text{Sets}$ covariant functor, \mathcal{R} finitely generated flat rings
- \mathcal{A}_X complex algebra
- evaluation maps: for all $x \in X(R)$, $\sigma : R \rightarrow \mathbb{C} \Rightarrow e_{x,\sigma} : \mathcal{A}_X \rightarrow \mathbb{C}$ algebra homomorphism

$$e_{f(y),\sigma} = e_{y,\sigma \circ f}$$

for $f : R' \rightarrow R$ ring homomorphism

Affine varieties $V_{\mathbb{Z}} \Rightarrow$ gadget $X = G(V_{\mathbb{Z}})$ with $X(R) = \text{Hom}(O(V), R)$ and $\mathcal{A}_X = O(V) \otimes \mathbb{C}$

Affine variety over \mathbb{F}_1 (Soulé)

Gadget with $X(R)$ finite; variety $X_{\mathbb{Z}}$ and morphism of gadgets

$$X \rightarrow G(X_{\mathbb{Z}})$$

such that all $X \rightarrow G(V_{\mathbb{Z}})$ come from $X_{\mathbb{Z}} \rightarrow V_{\mathbb{Z}}$

Should think: $R = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \Rightarrow X(R)$ cyclotomic points

Λ -rings, endomotives, and \mathbb{F}_1

J. Borger, *Λ -rings and the field with one element*, preprint 2009

- Grothendieck: characteristic classes, Riemann–Roch

- Λ -ring structure: “descent data” for a ring from \mathbb{Z} to \mathbb{F}_1

Torsion free R with action of semigroup \mathbb{N} by endomorphisms lifting Frobenius

$$s_p(x) - x^p \in pR, \quad \forall x \in R$$

Morphisms: $f \circ s_k = s_k \circ f$

\mathbb{Q} -algebra $A \Rightarrow \Lambda$ -ring

iff action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}$ on $\mathcal{X} = \text{Hom}(A, \bar{\mathbb{Q}})$ factors through an action of $\hat{\mathbb{Z}}$

NCG of \mathbb{Q} -lattices

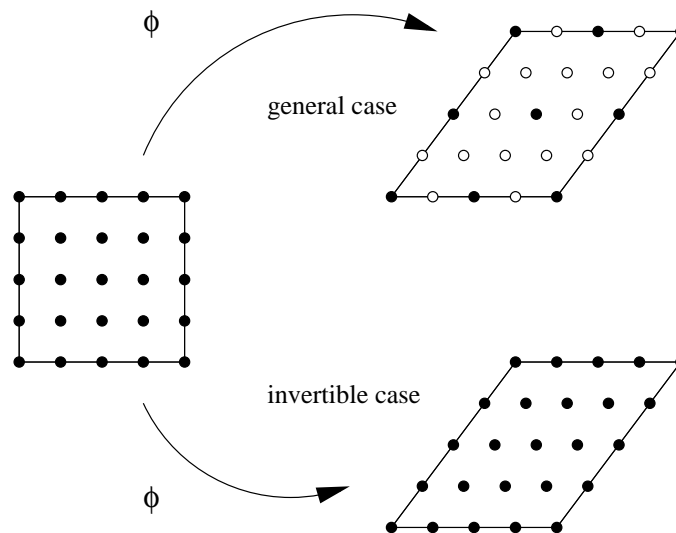
Alain Connes, M.M., *Quantum statistical mechanics of \mathbb{Q} -lattices* in “Frontiers in number theory, physics, and geometry. I”, 269–347, Springer, Berlin, 2006.

Definition: (Λ, ϕ) \mathbb{Q} -lattice in \mathbb{R}^n

lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points

$$\phi : \mathbb{Q}^n / \mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda / \Lambda$$

group *homomorphism* (invertible \mathbb{Q} -lat if isom)



Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$

iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \pmod{\Lambda_1 + \Lambda_2}$

\mathbb{Q} -lattices / Commensurability \Rightarrow NC space

1-dimensional case

$$(\Lambda, \phi) = (\lambda \mathbb{Z}, \lambda \rho) \quad \lambda > 0$$

$$\rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$$

Up to scaling λ : algebra $C(\hat{\mathbb{Z}})$

Commensurability Action of $\mathbb{N} = \mathbb{Z}_{>0}$

$$\alpha_n(f)(\rho) = f(n^{-1}\rho) \quad \text{zero otherwise}$$

(partially defined action of \mathbb{Q}_+^*)

1-dimensional \mathbb{Q} -lattices up to scale / Commens.

$$\Rightarrow \text{NC space } C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$$

Crossed product algebra

$$f_1 * f_2(r, \rho) = \sum_{s \in \mathbb{Q}_+^*, s\rho \in \hat{\mathbb{Z}}} f_1(rs^{-1}, s\rho) f_2(s, \rho)$$

$$f^*(r, \rho) = \overline{f(r^{-1}, r\rho)}$$

Bost–Connes algebra

J.B. Bost, A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*. *Selecta Math.* (N.S.) Vol.1 (1995) N.3, 411–457.

Algebra $\mathcal{A}_{\mathbb{Q},BC} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ generators and relations

$$\begin{aligned}\mu_n \mu_m &= \mu_{nm} \\ \mu_n \mu_m^* &= \mu_m^* \mu_n \quad \text{when } (n, m) = 1 \\ \mu_n^* \mu_n &= 1\end{aligned}$$

$$e(r + s) = e(r)e(s), \quad e(0) = 1$$

$$\rho_n(e(r)) = \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

C^* -algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\widehat{\mathbb{Z}}) \rtimes \mathbb{N}$

Time evolution

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it} \mu_n$$

Hamiltonian $\text{Tr}(e^{-\beta H}) = \zeta(\beta)$

Quantum statistical mechanics

(\mathcal{A}, σ_t) C^* -algebra and time evolution

State: $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ linear $\varphi(1) = 1$, $\varphi(a^*a) \geq 0$

Time evolution $\sigma_t \in \text{Aut}(\mathcal{A})$

rep. on Hilbert space $\mathcal{H} \Rightarrow$ Hamiltonian $H = \frac{d}{dt}\sigma_t|_{t=0}$

Equilibrium states (inverse temperature $\beta = 1/kT$)

$$\frac{1}{Z(\beta)} \text{Tr}(a e^{-\beta H}) \quad Z(\beta) = \text{Tr}(e^{-\beta H})$$

Classical points of NC space

KMS states $\varphi \in \text{KMS}_\beta$ ($0 < \beta < \infty$)

$\forall a, b \in \mathcal{A} \exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$

More on Bost–Connes

- Representations π_ρ on $\ell^2(N)$:

$$\mu_n \epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r)) \epsilon_m = \zeta_r^m \epsilon_m$$

$\zeta_r = \rho(e(r))$ root of 1, for $\rho \in \hat{\mathbb{Z}}^*$

- Low temperature extremal KMS ($\beta > 1$)

$$\varphi_{\beta,\rho}(a) = \frac{\text{Tr}(\pi_\rho(a)e^{-\beta H})}{\text{Tr}(e^{-\beta H})}, \quad \rho \in \hat{\mathbb{Z}}^*$$

High temperature: unique KMS state

- Zero temperature: evaluations $\varphi_{\infty,\rho}(e(r)) = \zeta_r$

$$\varphi_{\infty,\rho}(a) = \langle \epsilon_1, \pi_\rho(a) \epsilon_1 \rangle$$

Intertwining: $a \in \mathcal{A}_{\mathbb{Q},BC}$, $\gamma \in \hat{\mathbb{Z}}^*$

$$\varphi_{\infty,\rho}(\gamma a) = \theta_\gamma(\varphi_{\infty,\rho}(a))$$

$$\theta : \hat{\mathbb{Z}}^* \xrightarrow{\cong} \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

Class field theory isomorphism

Bost–Connes endomotive

A. Connes, C. Consani, M.M., *Noncommutative geometry and motives: the thermodynamics of endomotives*, Adv. in Math. 214 (2) (2007), 761–831

$A = \varinjlim_n A_n$ with $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$
abelian semigroup action $S = \mathbb{N}$ on $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

Endomotives (A, S) from self maps of algebraic varieties $s : Y \rightarrow Y$, $s(y_0) = y_0$ unbranched,
 $X_s = s^{-1}(y_0)$, $X = \varprojlim X_s = \text{Spec}(A)$

$$\xi_{s,s'} : X_{s'} \rightarrow X_s, \quad \xi_{s,s'}(y) = r(y), \quad s' = rs \in S$$

Bost–Connes endomotive:

\mathbb{G}_m with self maps $u \mapsto u^k$

$$s_k : P(t, t^{-1}) \mapsto P(t^k, t^{-k}), \quad k \in \mathbb{N}, \quad P \in \mathbb{Q}[t, t^{-1}]$$

$$\xi_{k,\ell}(u(\ell)) = u(\ell)^{k/\ell}, \quad u(\ell) = t \pmod{t^\ell - 1}$$

$X_k = \text{Spec}(\mathbb{Q}[t, t^{-1}]/(t^k - 1)) = s_k^{-1}(1)$ and
 $X = \varprojlim_k X_k$

$$u(\ell) \mapsto e(1/\ell) \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}], \quad C(X(\bar{\mathbb{Q}})) = C(\hat{\mathbb{Z}})$$

Integer model of the Bost–Connes algebra

A. Connes, C. Consani, M.M., *Fun with \mathbb{F}_1* , to appear in
 J. Number Theory, arXiv:0806.2401

$\mathcal{A}_{\mathbb{Z}, BC}$ generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and $\mu_n^*, \tilde{\mu}_n$

$$\begin{aligned}\tilde{\mu}_n \tilde{\mu}_m &= \tilde{\mu}_{nm} \\ \mu_n^* \mu_m^* &= \mu_{nm}^* \\ \mu_n^* \tilde{\mu}_n &= n \\ \tilde{\mu}_n \mu_m^* &= \mu_m^* \tilde{\mu}_n \quad (n, m) = 1.\end{aligned}$$

$$\mu_n^* x = \sigma_n(x) \mu_n^* \quad \text{and} \quad x \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x)$$

where $\sigma_n(e(r)) = e(nr)$ for $r \in \mathbb{Q}/\mathbb{Z}$

Note: $\rho_n(x) = \mu_n x \mu_n^*$ ring homomorphism but
 not $\tilde{\rho}_n(x) = \tilde{\mu}_n x \mu_n^*$ (correspondences “crossed
 product” $\mathcal{A}_{\mathbb{Z}, BC} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$)

NCG and Soulé's \mathbb{F}_1 -geometry:

Roots of unity as varieties over \mathbb{F}_1 :

$$\underline{\mu}^{(k)}(R) = \{x \in R \mid x^k = 1\} = \text{Hom}_{\mathbb{Z}}(A_k, R)$$

$$A_k = \mathbb{Z}[t, t^{-1}] / (t^k - 1)$$

- Inductive system \mathbb{G}_m over \mathbb{F}_1 :

$$\underline{\mu}^{(n)}(R) \subset \underline{\mu}^{(m)}(R), \quad n|m, \quad A_m \twoheadrightarrow A_n$$

$$\mathcal{A}_X = C(S^1)$$

- Projective system (BC): $\xi_{m,n} : X_n \twoheadrightarrow X_m$

$$\xi_{m,n} : \underline{\mu}^{(n)}(R) \twoheadrightarrow \underline{\mu}^{(m)}(R), \quad n|m$$

$$\underline{\mu}^{\infty}(R) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R)$$

\Rightarrow projective system of affine varieties over \mathbb{F}_1

$$\xi_{m,n} : \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} \rightarrow \mathbb{F}_{1^m} \otimes_{\mathbb{F}_1} \mathbb{Z}$$

$$\mathcal{A}_X = \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$$

Bost–Connes and \mathbb{F}_1

A. Connes, C. Consani, M.M., *Fun with \mathbb{F}_1* , arXiv:0806.2401

Affine varieties $\mu^{(n)}$ over \mathbb{F}_1 defined by gadgets $G(\text{Spec}(\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]))$; projective system

Endomorphisms σ_n (of varieties over \mathbb{Z} , of gadgets, of \mathbb{F}_1 -varieties)

Extensions \mathbb{F}_{1n} : free actions of roots of 1
(Kapranov–Smirnov)

$$\zeta \mapsto \zeta^n, \quad n \in \mathbb{N} \quad \text{and} \quad \zeta \mapsto \zeta^\alpha \leftrightarrow e(\alpha(r)), \quad \alpha \in \hat{\mathbb{Z}}$$

Frobenius action on $\mathbb{F}_{1\infty}$

In reductions mod p of integral Bost–Connes endomotive \Rightarrow Frobenius

Bost–Connes = extensions \mathbb{F}_{1n} plus Frobenius

Characteristic p versions of the BC endomotive

$$\mathbb{Q}/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$$

denom = power of p ; denom = prime to p

$$\mathbb{K}[\mathbb{Q}_p/\mathbb{Z}_p] \rtimes p^{\mathbb{Z}^+}$$

endomorphisms σ_n for $n = p^\ell$, $\ell \in \mathbb{Z}^+$

$\varphi_{\mathbb{F}_p}(x) = x^p$ Frobenius of \mathbb{K} char p

$$(\sigma_{p^\ell} \otimes \varphi_{\mathbb{F}_p}^\ell)(f) = f^{p^\ell}$$

$f \in \mathbb{K}[\mathbb{Q}/\mathbb{Z}]$

$$(\sigma_{p^\ell} \otimes \varphi_{\mathbb{F}_p}^\ell)(e(r) \otimes x) = e(p^\ell r) \otimes x^{p^\ell} = (e(r) \otimes x)^{p^\ell}$$

\Rightarrow BC endomorphisms restrict to Frobenius on mod p reductions: σ_{p^ℓ} Frobenius correspondence on pro-variety $\mu^\infty \otimes_{\mathbb{Z}} \mathbb{K}$

NCG and Borger's \mathbb{F}_1 -geometry

M.M., *Cyclotomy and endomotives*, preprint arXiv:0901.3167

Multivariable Bost–Connes endomotives

Variety $\mathbb{T}^n = (\mathbb{G}_m)^n$ endomorphisms $\alpha \in M_n(\mathbb{Z})^+$

$$X_\alpha = \{t = (t_1, \dots, t_n) \in \mathbb{T}^n \mid s_\alpha(t) = t_0\}$$

$$\xi_{\alpha, \beta} : X_\beta \rightarrow X_\alpha, \quad t \mapsto t^\gamma, \quad \alpha = \beta\gamma \in M_n(\mathbb{Z})^+$$

$$t \mapsto t^\gamma = \sigma_\gamma(t) = (t_1^{\gamma_{11}} t_2^{\gamma_{12}} \cdots t_n^{\gamma_{1n}}, \dots, t_1^{\gamma_{n1}} t_2^{\gamma_{n2}} \cdots t_n^{\gamma_{nn}})$$

$X = \varprojlim_\alpha X_\alpha$ with semigroup action

$C(X(\bar{\mathbb{Q}})) \cong \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ generators $e(r_1) \otimes \cdots \otimes e(r_n)$

$$\mathcal{A}_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+$$

generated by $e(\underline{r})$ and μ_α, μ_α^*

$$\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r}) \mu_\alpha^* = \frac{1}{\det \alpha} \sum_{\alpha(\underline{s}) = \underline{r}} e(\underline{s})$$

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha = e(\alpha(\underline{r}))$$

Endomorphisms:

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha$$

The Bost–Connes endomotive is a direct limit of Λ -rings

$$R_n = \mathbb{Z}[t, t^{-1}]/(t^n - 1) \quad s_k(P)(t, t^{-1}) = P(t^k, t^{-k})$$

Action of $\hat{\mathbb{Z}}$:

$$\alpha \in \hat{\mathbb{Z}} : \quad (\zeta : x \mapsto \zeta x) \mapsto (\zeta : x \mapsto \zeta^\alpha x)$$

$$\hat{\mathbb{Z}} = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

\Rightarrow Frobenius over \mathbb{F}_{1^∞} (Haran)

Embeddings of Λ -rings (Borger–de Smit)

Every torsion free finite rank Λ -ring embeds in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ with action of \mathbb{N} compatible with $S_{n,diag} \subset M_n(\mathbb{Z})^+$

BC systems as universal Λ -rings

\Rightarrow multivariable Bost–Connes endomotives as universal Λ -rings

Quantum statistical mechanics of multivariable Bost–Connes endomotives

Representations on $\ell^2(M_n(\mathbb{Z})^+)$: $\zeta_i^\alpha = \prod_j \zeta_i^{\alpha_{ij}}$

$$\zeta = (\zeta_1, \dots, \zeta_n) = \rho(\underline{r}), \quad \rho \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$$

$$\mu_\beta \epsilon_\alpha = \epsilon_{\beta\alpha}$$

$$\pi_\rho(e(\underline{r}))\epsilon_\alpha = \prod_i \zeta_i^{\alpha^T} \epsilon_\alpha$$

Time evolution:

$$\sigma_t(e(\underline{r})) = e(\underline{r}), \quad \sigma_t(\mu_\alpha) = \det(\alpha)^{it} \mu_\alpha$$

Hamiltonian:

$$H\epsilon_\alpha = \log \det(\alpha) \epsilon_\alpha$$

Problem: Infinite multiplicities in the spectrum:
 $\mathrm{SL}_n(\mathbb{Z})$ -symmetry

Groupoid and convolution algebra

Similar to: A.Connes, M.M. *Quantum statistical mechanics of \mathbb{Q} -lattices*, in “Frontiers in Number Theory, Physics, and Geometry, I” pp.269–350, Springer, 2006.

$$\mathcal{U}_\Gamma = \{(\alpha, \rho) \in \Gamma \backslash \mathrm{GL}_n(\mathbb{Q})^+ \times_\Gamma \widehat{\mathbb{Z}}^n \mid \alpha(\rho) \in \widehat{\mathbb{Z}}^n\}$$

Quotient \mathcal{U}_Γ by $\mathrm{SL}_n(\mathbb{Z}) \times \mathrm{SL}_n(\mathbb{Z})$

$$(\gamma_1, \gamma_2) : (\alpha, \rho) \mapsto (\gamma_1 \alpha \gamma_2^{-1}, \gamma_2(\rho))$$

Convolution algebra

$$(f_1 \star f_2)(\alpha, \rho) = \sum_{(\alpha, \rho) = (\alpha_1, \rho_1) \circ (\alpha_2, \rho_2) \in \mathcal{U}_\Gamma} f_1(\alpha_1, \rho_1) f_2(\alpha_2, \rho_2)$$

$$f^*(\alpha, \rho) = \overline{f(\alpha^{-1}, \alpha(\rho))} \quad \text{and} \quad \sigma_t(f)(\alpha, \rho) = \det(\alpha)^{it} f(\alpha, \rho)$$

$$(\pi_\rho(f)\xi)(\alpha) = \sum_{\beta \in \Gamma \backslash \mathrm{GL}_n(\mathbb{Q})^+ : \beta\rho \in \widehat{\mathbb{Z}}^*} f(\alpha\beta^{-1}, \beta(\rho))\xi(\beta)$$

On $\ell^2(\Gamma \backslash G_\rho)$. If $\rho \in (\widehat{\mathbb{Z}}^*)^n$:

$$G_\rho = \{\alpha \in \mathrm{GL}_n(\mathbb{Q})^+ \mid \alpha(\rho) \in \widehat{\mathbb{Z}}^n\} = M_n(\mathbb{Z})^+$$

$$Z(\beta) = \sum_{m \in \Gamma \backslash M_n(\mathbb{Z})^+} \det(m)^{-\beta}$$

Why QSM?

A. Connes, C. Consani, M.M., *The Weil proof and the geometry of the adèles class space*, arXiv:math/0703392

For original BC system \Rightarrow dual system

$\hat{A} = \mathcal{A} \rtimes_{\sigma} \mathbb{R}$. Restriction map:

$$\delta : \hat{\mathcal{A}}_{\beta}^{\natural} \rightarrow \mathcal{S}^{\natural}(\Omega_{\beta})$$

Cyclic module: cokernel $HC_0(D(\mathcal{A}, \varphi))$

\Rightarrow Spectral realization of zeros of Riemann zeta function and trace formula

Multivariable case:

Manin (1994): problem of $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} = ???$

cf. Weil's proof for function fields $C \times_{\mathbb{F}_q} C$

Zeta function viewpoint: Kurokawa

tensor product zeta function: summing over zeros

$$\prod_{\lambda \in \Xi} (s - \lambda)^{m_{\lambda}} \otimes \prod_{\mu \in \Theta} (s - \mu)^{n_{\mu}} := \prod_{(\lambda, \mu)} (s - \lambda - \mu)^{m_{\lambda} + n_{\mu}}$$

as zeta-regularized infinite products

Question: Kurokawa zeta functions from multivariable dual systems?

Thermodynamics of endomotives and RH

A. Connes, C. Consani, M.M., *Noncommutative geometry and motives: the thermodynamics of endomotives*, Adv. in Math. 214 (2) (2007), 761–831

- Dual system: $\hat{\mathcal{A}} = \mathcal{A}_{BC} \rtimes_{\sigma} \mathbb{R}$
- Scaling action: $\theta_{\lambda}(\int x(t) U_t dt) = \int \lambda^{it} x(t) U_t dt$
- Classical points: $\tilde{\Omega}_{\beta} \simeq \Omega_{\beta} \times \mathbb{R}_{+}^{*}$ low T KMS states
- Restriction map: (morphism of cyclic modules)

$$\delta : \hat{\mathcal{A}}_{\beta} \xrightarrow{\pi} C(\tilde{\Omega}_{\beta}, \mathcal{L}^1) \xrightarrow{\text{Tr}} C(\tilde{\Omega}_{\beta})$$

- Cokernel (abelian category) $D(\mathcal{A}, \varphi) = \text{Coker}(\delta)$
- Cyclic homology (with scaling action) $HC_0(D(\mathcal{A}, \varphi))$
- Galois action + scaling $C_{\mathbb{Q}} = \hat{\mathbb{Z}}^{*} \times \mathbb{R}_{+}^{*}$

Weil explicit formula as trace formula on $\mathcal{H}^1 = HC_0(D(\mathcal{A}, \varphi))$:

$$\text{Tr}(\vartheta(f)|_{\mathcal{H}^1}) = \hat{f}(0) + \hat{f}(1) - \Delta \bullet \Delta f(1) - \sum_v \int'_{(\mathbb{K}_v^*, e_{K_v})} \frac{f(u^{-1})}{|1-u|} d^*u$$

$$\vartheta(f) = \int_{C_{\mathbb{Q}}} f(g) \vartheta_g d^*g \quad f \in S(C_{\mathbb{Q}})$$

Self inters of diagonal $\Delta \bullet \Delta = \log |a| = -\log |D|$

($D =$ discriminant for $\#$ -field, Euler char $\chi(C)$ for $\mathbb{F}_q(C)$)

Note:

- Connes' 1997 RH paper: $\text{Tr}(R_\Lambda U(f))$:
 - only critical line zeros
 - Trace formula (global) \Leftrightarrow RH
- $\text{Tr}(\vartheta(f)|_{\mathcal{H}^1})$:
 - all zeros involved
 - RH \Leftrightarrow positivity

$$\text{Tr}(\vartheta(f \star f^\sharp)|_{\mathcal{H}^1}) \geq 0 \quad \forall f \in S(C_{\mathbb{Q}})$$

where

$$(f_1 \star f_2)(g) = \int f_1(k) f_2(k^{-1}g) d^*g$$

multiplicative Haar measure d^*g and adjoint

$$f^\sharp(g) = |g|^{-1} \overline{f(g^{-1})}$$

\Rightarrow Better for comparing with Weil's proof for function fields

Weil's proof in a nutshell

$\mathbb{K} = \mathbb{F}_q(C)$ function field, $\Sigma_{\mathbb{K}} =$ places $\deg n_v = \#$ orbit of Fr on fiber $C(\bar{\mathbb{F}}_q) \rightarrow \Sigma_{\mathbb{K}}$

$$\zeta_{\mathbb{K}}(s) = \prod_{\Sigma_{\mathbb{K}}} (1 - q^{-n_v s})^{-1} = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

$P(T) = \prod (1 - \lambda_n T)$ char polynomial of Fr* on $H_{\text{et}}^1(\bar{C}, \mathbb{Q}_\ell)$

$$C(\bar{\mathbb{F}}_q) \supset \text{Fix}(\text{Fr}^j) = \sum_k (-1)^k \text{Tr}(\text{Fr}^{*j} | H_{\text{et}}^k(\bar{C}, \mathbb{Q}_\ell))$$

RH \Leftrightarrow eigenvalues λ_n with $|\lambda_j| = q^{1/2}$

Correspondences: divisors $Z \subset C \times C$; degree, codegree, trace:

$$d(Z) = Z \bullet (P \times C) \quad d'(Z) = Z \bullet (C \times P)$$

$$\text{Tr}(Z) = d(Z) + d'(Z) - Z \bullet \Delta$$

RH \Leftrightarrow Weil positivity $\text{Tr}(Z \star Z') > 0$

\Rightarrow Dictionary between NCG and Alg Geom:

$$C \times_{\mathbb{F}_q} C \Leftrightarrow \text{NCG version of } \text{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \text{Spec}(\mathbb{Z})$$

A possible intermediate step: Witt vectors

Affine ring scheme of big Witt vectors

$$\mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_n, \dots]$$

symmetric functions λ_i , ghost coordinates

$$\psi_n = \sum_{d|n} d u_d^{n/d}, \quad \text{with} \quad \psi_n = x_1^n + x_2^n + \dots$$

$W(R) = \prod_{n \geq 1} R$ addition/multipl of ghost coords

Truncated Witt schemes with $\mathbb{Z}[u_1, \dots, u_N] \Rightarrow$
gadgets $\mathcal{W}_{\mathbb{Z}}^{(N)}$ over \mathbb{F}_1 (Manin)

Also Λ -ring structure given by the ψ_n

Borger's suggestion: Witt schemes related to

$$\text{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \cdots \times_{\mathbb{F}_1} \text{Spec}(\mathbb{Z})$$

\Rightarrow Realize inside multivariable BC endomotives

Other aspects of \mathbb{F}_1 -geometry:

Analytic geometry

Yu.I. Manin, *Cyclotomy and analytic geometry over \mathbb{F}_1* ,
arXiv:0809.1564.

The Habiro ring

K. Habiro, *Cyclotomic completions of polynomial rings*,
Publ. RIMS Kyoto Univ. (2004) Vol.40, 1127–1146.

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q)_n)$$

$$(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$$

$\mathbb{Z}[q]/((q)_n) \twoheadrightarrow \mathbb{Z}[q]/((q)_k)$ for $k \leq n$ since $(q)_k | (q)_n$

Evaluation maps at roots of 1: surj ring homom

$$ev_\zeta : \widehat{\mathbb{Z}[q]} \rightarrow \mathbb{Z}[\zeta]$$

give an *injective* homomorphism:

$$ev : \widehat{\mathbb{Z}[q]} \rightarrow \prod_{\zeta \in \mathcal{Z}} \mathbb{Z}[\zeta]$$

Taylor series expansions

$$\tau_\zeta : \widehat{\mathbb{Z}[q]} \rightarrow \mathbb{Z}[\zeta][[q - \zeta]]$$

injective ring homomorphism

Ring of “analytic functions on roots of unity”
 \Rightarrow Another model for the NCG of the cyclotomic tower, replacing $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ with $\widehat{\mathbb{Z}[q]}$

Multivariable Habiro rings (Manin)

$$\mathbb{Z}[\widehat{q_1, \dots, q_n}] = \varprojlim_N \mathbb{Z}[q_1, \dots, q_n] / I_{n,N}$$

where $I_{n,N}$ is the ideal

$$((q_1 - 1)(q_1^2 - 1) \cdots (q_1^N - 1), \dots, (q_n - 1)(q_n^2 - 1) \cdots (q_n^N - 1))$$

Evaluations at roots of 1

$$ev_{(\zeta_1, \dots, \zeta_n)} : \mathbb{Z}[\widehat{q_1, \dots, q_n}] \rightarrow \mathbb{Z}[\zeta_1, \dots, \zeta_n]$$

Taylor expansions

$$T_Z : \mathbb{Z}[\widehat{q_1, \dots, q_n}] \rightarrow \mathbb{Z}[\zeta_1, \dots, \zeta_n][[q_1 - \zeta_1, \dots, q_n - \zeta_n]]$$

$$Z = (\zeta_1, \dots, \zeta_n) \text{ in } \mathcal{Z}^n$$

Endomotives: Habiro ring version

M.M., *Cyclotomy and endomotives*, preprint arXiv:0901.3167

One variable:

$$\sigma_n(f)(q) = f(q^n)$$

lifts $P(\zeta) \mapsto P(\zeta^n)$ in $\mathbb{Z}[\zeta]$ through ev_ζ

Action of $\mathbb{N} \Rightarrow$ group crossed product

$$\mathcal{A}_{\mathbb{Z},q} = \widehat{\mathbb{Z}[q]}_\infty \rtimes \mathbb{Q}_+^*$$

$\mathcal{A}_{\mathbb{Z},q}$ = generated by $\widehat{\mathbb{Z}[q]}$ and μ_n, μ_n^*

$$\mu_n \sigma_n(f) = f \mu_n, \quad \mu_n^* f = \sigma_n(f) \mu_n^*$$

$\widehat{\mathbb{Z}[q]}_\infty = \cup_N A_N$ with A_N gen by $\mu_N f \mu_N^*$

$$\widehat{\mathbb{Z}[q]}_\infty = \varinjlim_n (\sigma_n : \widehat{\mathbb{Z}[q]} \rightarrow \widehat{\mathbb{Z}[q]})$$

injective \Rightarrow automorphisms $\sigma_n : \widehat{\mathbb{Z}[q]}_\infty \rightarrow \widehat{\mathbb{Z}[q]}_\infty$

$\mathcal{P}_{\mathbb{Z}} = \text{polyn in } \mathbb{Q} \text{ powers } q^r$

$$\widehat{\mathcal{P}}_{\mathbb{Z}} = \varprojlim_N \mathcal{P}_{\mathbb{Z}} / \mathcal{J}_N$$

$\mathcal{J}_N = \text{ideal gen by } (q^r)_N = (1 - q^r) \cdots (1 - q^{rN}), r \in \mathbb{Q}_+^*$

$$\widehat{\mathbb{Z}[q]}_{\infty} \simeq \widehat{\mathcal{P}}_{\mathbb{Z}}, \quad \mu_n f \mu_n^* \mapsto f(q^{1/n})$$

$$\rho_r(f)(q) = f(q^r)$$

Multivariable:

$$\widehat{\mathbb{Z}[q_1, \dots, q_n]} = \varprojlim_N \mathbb{Z}[q_1, \dots, q_n, q_1^{-1}, \dots, q_n^{-1}] / \mathcal{J}_{n,N}$$

$\mathcal{J}_{n,N}$ ideal generated by the $(q_i - 1) \cdots (q_i^N - 1)$, for $i = 1, \dots, n$ and the $(q_i^{-1} - 1) \cdots (q_i^{-N} - 1)$

Torus $\mathbb{T}^n = (\mathbb{G}_m)^n$, algebra $\mathbb{Q}[t_i, t_i^{-1}]$

$$t^\alpha = (t_i^\alpha)_{i=1, \dots, n} \quad \text{with} \quad t_i^\alpha = \prod_j t_j^{\alpha_{ij}}$$

Semigroup action $\alpha \in M_n(\mathbb{Z})^+$:

$$q \mapsto \sigma_\alpha(q) = \sigma_\alpha(q_1, \dots, q_n) =$$

$$(q_1^{\alpha_{11}} q_2^{\alpha_{12}} \cdots q_n^{\alpha_{1n}}, \dots, q_1^{\alpha_{n1}} q_2^{\alpha_{n2}} \cdots q_n^{\alpha_{nn}}) = q^\alpha$$

Question: 3-manifolds and \mathbb{F}_1 -geometry?

Universal Witten–Reshetikhin–Turaev invariant

K.Habiro, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, Invent. Math. 171 (2008) 1–81

- Chern–Simons path integral (Witten)
- quantum groups at roots of 1 (Reshetikhin–Turaev)

$$\tau(M) : \mathcal{Z} \rightarrow \mathbb{C}, \quad \tau_\zeta(M)$$

- Ohtsuki series

$$\tau^O(M) = 1 + \sum_{n=1}^{\infty} \lambda_n(M)(q-1)^n$$

Unified view (Habiro): $J_M(q) = J_L(q)$

$$J_M(q) \in \widehat{\mathbb{Z}[q]}$$

function in the (one-variable) Habiro ring:

$$ev_\zeta(J_M(q)) = \tau_\zeta(M)$$

$$\tau_1(J_M(q)) = \tau^O(M)$$

Using: 3-dimensional integral homology sphere M
surgery presentation $M = S_L^3$, algebraically split link
 $L = L_1 \cup \cdots \cup L_\ell$ in S^3 framing ± 1

$$S_L^3 \cong S_{L'}^3 \Leftrightarrow L \sim L' \text{ Fenn–Rourke moves}$$

Integral homology 3-spheres

$\mathbb{Z}hs$ = free ab group generated by orientation-preserving homeomorphism classes of integral homology 3-spheres

Ring with product $M_1 \# M_2$ connected sum

$$J_{M_1 \# M_2}(q) = J_{M_1}(q)J_{M_2}(q), \quad J_{S^3}(q) = 1$$

$$J_{-M}(q) = J_M(q^{-1})$$

\Rightarrow WRT ring homomorphism

$$J : \mathbb{Z}hs \rightarrow \widehat{\mathbb{Z}[q]}$$

Ohtsuki filtration

$$\mathbb{Z}hs = F_0 \supset F_1 \supset \cdots F_k \supset \cdots$$

F_k \mathbb{Z} -submodule spanned by

$$[M, L_1, \dots, L_k] = \sum_{L' \subset \{L_1, \dots, L_k\}} (-1)^{|L'|} M_{L'}$$

L_i = alg split links ± 1 -framed

Habiro conjecture

$$J : \widehat{\mathbb{Z}hs} \rightarrow \widehat{\mathbb{Z}[q]}$$

$\widehat{\mathbb{Z}hs} = \varprojlim_d \mathbb{Z}hs / F_d$ with $d : \mathbb{N} \rightarrow \mathbb{Z}_{\geq 0}$

$d(n)$ -components of link have framing $\pm n$

Integral homology spheres and \mathbb{F}_1 ?

$$X_{\mathbb{Z}hs}(R) := \{\phi : \mathbb{Z}hs \rightarrow R \mid \exists \tilde{\phi} : \widehat{\mathbb{Z}[q]} \rightarrow R, \phi = \tilde{\phi} \circ J\}$$

$J : \mathbb{Z}hs \rightarrow \widehat{\mathbb{Z}[q]}$ WRT invariant;

$X_{\mathbb{Z}hs}(R)$ = set of “coarser” R -valued invariants

$\mathcal{A}_X := \mathbb{Z}hs \otimes \mathbb{C}$ WRT evaluations as cyclotomic points: $\sigma \circ \phi : \mathbb{Z}hs \rightarrow \mathbb{C}$ factors through some evaluation ev_ζ

$\Rightarrow X_{\mathbb{Z}hs}$ gadget over \mathbb{F}_1

Question: Using Habiro conjecture $X_{\widehat{\mathbb{Z}hs}}$ inductive limit of affine varieties over \mathbb{F}_1 ? (finiteness)

Question: 3-manifolds and endomotives?

Semigroup action? Is it possible to construct

$$M = S^3_{(L,m)} \mapsto \sigma_n(M) = S^3_{(L_n,m_n)}$$

(inv. under Fenn–Rourke moves)

$$\sigma_n(M_1 \# M_2) = \sigma_n(M_1) \sigma_n(M_2)$$

Question: $\exists ? \Upsilon_M \in \widehat{\mathbb{Z}[q]}$

$$\Upsilon_{\sigma_n(M)} = \sigma_n(\Upsilon_M)$$

Is it possible to use endomotives (eg multivariable BC) to construct 3-manifold invariants ?

$$\mathbb{Z}hs \rtimes \mathbb{N} \rightarrow \mathcal{A} = A \rtimes \mathbb{N}$$

Quantum statistical mechanics?

Categorical approach:

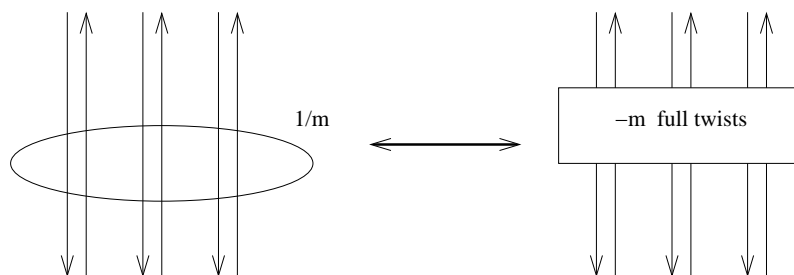
- Reidemeister moves

(link diagrams: same ambient isotopy class)

- Hoste moves (admissible links)

(same $\mathbb{Z}h_s$: trivial linking #s and frames $1/m$)

component $1/m$ -framing replaced by $-m$ full twists



2-category:

- Objects: (D, \mathfrak{r}) admissible framed link diagrams
- 1-morphisms: $(D, \mathfrak{r}) \xrightarrow{H} (D', \mathfrak{r}')$ Hoste moves
- 2-morphisms: $\mathcal{R} = (R, R')$ Reidemeister moves

$$\begin{array}{ccc} (D_1, \mathfrak{r}_1) & \xrightarrow{H_1} & (D'_1, \mathfrak{r}'_1) \\ \downarrow R & & \downarrow R' \\ (D_2, \mathfrak{r}_2) & \xrightarrow{H_2} & (D'_2, \mathfrak{r}'_2) \end{array}$$

Convolution algebra: $\mathcal{A}_{\mathcal{CL}(2)}$ functions with finite support on 2-Mor

- vertical convolution product

$$(f_1 \circ f_2)(\mathcal{R}) = \sum_{\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2} f_1(\mathcal{R}_1) f_2(\mathcal{R}_2)$$

- horizontal convolution product

$$(f_1 \bullet f_2)(\mathcal{R}) = \sum_{\mathcal{R} = \mathcal{R}_1 \bullet \mathcal{R}_2} f_1(\mathcal{R}_1) f_2(\mathcal{R}_2)$$

For compositions of 2-morphisms:

- horizontal $\mathcal{R}_1 \bullet \mathcal{R}_2$ with $\mathcal{R}_1 = (R, R')$ and $\mathcal{R}_2 = (R', R'')$

$$\begin{array}{ccccc}
 (D_1, \mathfrak{r}_1) & \xrightarrow{H_1} & (D'_1, \mathfrak{r}'_1) & \xrightarrow{H'_1} & (D''_1, \mathfrak{r}''_1) \\
 \downarrow R & & \downarrow R' & & \downarrow R'' \\
 (D_2, \mathfrak{r}_2) & \xrightarrow{H_2} & (D'_2, \mathfrak{r}'_2) & \xrightarrow{H'_2} & (D''_2, \mathfrak{r}''_2)
 \end{array}$$

- vertical $\mathcal{R}_1 \circ \mathcal{R}_2$ with $\mathcal{R}_1 = (R_1, R'_1)$ and $\mathcal{R}_2 = (R_2, R'_2)$

$$\begin{array}{ccc}
 (D_1, \mathfrak{r}_1) & \xrightarrow{H_1} & (D'_1, \mathfrak{r}'_1) \\
 \downarrow R_1 & & \downarrow R'_1 \\
 (D_2, \mathfrak{r}_2) & \xrightarrow{H_2} & (D'_2, \mathfrak{r}'_2) \\
 \downarrow R_2 & & \downarrow R'_2 \\
 (D_3, \mathfrak{r}_3) & \xrightarrow{H_3} & (D'_3, \mathfrak{r}'_3)
 \end{array}$$

Involution: $f^*(\mathcal{R}) = \overline{f(\mathcal{R}^{-1})}$

$$\begin{array}{ccc} (D'_2, \mathbf{r}'_2) & \xrightarrow{H_2^{-1}} & (D_2, \mathbf{r}_2) \\ \downarrow R'^{-1} & & \downarrow R^{-1} \\ (D'_1, \mathbf{r}'_1) & \xrightarrow{H_1^{-1}} & (D_1, \mathbf{r}_1) \end{array}$$

- Horizontal time evolution: $\sigma_f(f_1 \circ f_2) = \sigma_t(f_1) \circ \sigma_t(f_2)$

$$\sigma_t(f)(H) = e^{imt} f(H), \quad \text{with} \quad m = \ell - \ell'$$

Hoste move between $L = L_1 \cup \dots \cup L_\ell$ and $L' = L'_1 \cup \dots \cup L'_\ell$

- Vertical time evolution: $\sigma_f(f_1 \bullet f_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)$
self-linking number with respect to the black-board framing of the link diagram

(changed by Reidemeister moves)

Variant: only oriented Hoste moves (semigroupoid)

Cabling: $L_n = c_n(L)$ with $\mathfrak{n} = (n, \dots, n)$

(L, \mathfrak{r}) admissible and $n|m_i$: if oriented Hoste move

$$(L, \mathfrak{r}) \xrightarrow{H} (L', \mathfrak{r}')$$

then $n|m_i$ also for L' and cabled oriented Hoste move

$$(L_n, n\mathfrak{r}) \xrightarrow{H_n} (L'_n, n\mathfrak{r}')$$

Action:

$$\rho_n(f)(H) = \begin{cases} f(H_n) & n|m_i, \forall i, \\ 0 & \text{otherwise,} \end{cases}$$

endomorphisms of $\mathcal{A}_{\mathcal{CL}(1)}$

(after collapsing 2-morphisms to equiv rel)

Similar to case of

M.M., A.Zainy al-Yasry, *Coverings, correspondences, and noncommutative geometry*, J.Geom.Phys. Vol.58 (2008) N.12, 1639–1661

2-category with Obj=embedded graphs,
1-Mor=3-manifolds as branched coverings,
2-Mor=4-dim cobordisms

Horizontal and vertical time evolutions

(multiplicity of coverings, index theorems, Hartle–Hawking gravity)

Further questions: Relation of the Habiro rings and (semi)group actions to Zagier’s quantum modular forms ?