MIDTERM TEST 3

MATH 204

In every problem, you should show your work, not just the final result. You should also explain the steps of your solution. Try to consider all possible cases.

Problem 1. Prove that if $v_1, ..., v_m$ are linearly dependent vectors in $\mathbb{R}^n$ and v_{m+1} is another vector in $\mathbb{R}^n$, then the vectors $v_1, ..., v_m, v_{m+1}$ are linearly dependent.

Solution: since $v_1, ..., v_m$ are linearly dependent, there exist numbers $\alpha_1, ..., \alpha_m$, not all $= 0$, such that $\alpha_1 v_1 + ... + \alpha_m v_m = 0$. Then $\alpha_1 v_1 + ... + \alpha_m v_m + 0 \cdot v_{m+1} = 0$ (i.e. let $\alpha_{m+1} = 0$. Since not all numbers $\alpha_1, ..., \alpha_{m+1}$ are zeroes, the vectors $v_1, ..., v_{m+1}$ are linearly dependent.

Problem 2. Let $T\colon V \to W$ be a linear transformation. Let $\{v_1, ..., v_n\}$ be a basis of V and suppose that $T(v_1) = 0, T(v_2) = 0, ..., T(v_n) = 0$. Show that for every v in V, $T(v) = 0$.

Solution: Let $v \in V$. Since $v_1, ..., v_n$ is a basis of V, $v = \alpha_1 v_1 + ... + \alpha_n v_n$ for some numbers $\alpha_1, ..., \alpha_n$. Then $T(v) = \alpha_1 T(v_1) + ... + \alpha_n T(v_n)$ (since T is a linear transformation), so $T(v) = 0$.

Problem 3. In $\mathbb{R}^5$, find a basis of the subspace

$$\{(x_1, x_2, x_3, x_4, x_5, x_6) \mid 2x_1 - 3x_2 + x_3 + 4x_4 - x_5 = 0\}.$$

Solution: We have

$$\begin{cases} x_1 = & x_1 & & & \\ x_2 = & & x_2 & & \\ x_3 = & & & x_3 & \\ x_4 = & & & & x_4 \\ x_5 = & 2x_1 - & 3x_2 + & x_3 + & 4x_4 \end{cases}$$

So a basis is $\{(1,0,0,0,2), (0,1,0,0,-3), (0,0,1,0,1), (0,0,0,1,4)\}$.

Problem 4. Find a basis for the span of the following vectors in $\mathbb{R}^4$:

$$(1,0,0,1), (0,1,2,3), (2,-3,-6,-8), (1,1,2,5).$$

Solution: Put these vectors as columns in a matrix, reduce the matrix to RREF. The first three columns will be pivotal. So a basis of the column space consist of the first three of the given vectors: $\{(1,0,0,1), (0,1,2,3), (2,-3,-6,-8)\}$. Another solution: put these vectors as rows in a matrix, reduce the matrix to a RREF. Non-zero rows of that RREF form a basis of the span.

Problem 5. a) Find an orthonormal basis of the solution space of the equation $2x - y + 3z - w = 0$ (in $\mathbb{R}^4$).

Solution: The subspace has a basis:

$$\{(1,2,0,0), (0,3,1,0), (0,-1,0,1)\}.$$

Applying the Gram-Schmidt to this basis, we obtain:

$$\left\{\frac{1}{\sqrt{5}}(1,2,0,0), \frac{1}{\sqrt{70}}(-6,3,5,0), \frac{1}{\sqrt{210}}(2,-1,3,14)\right\}$$

b)Find the projection of the vector $(1,-1,2,3)$ onto that subspace.

Solution: The formula: $pr_H(u) = (u \cdot e_1)e_1 + ... + (u \cdot e_m)e_m$ where $e_1, ..., e_m$ is an orthonormal basis of a subspace H. So using the basis from part a), we get the answer:

$$(1/5, -3/5, 4/5, 17/5).$$

Another solution: The vector $n = (2,-1,3,-1)$ spans the orthogonal compliment of the subspace H in question. So the projection onto the subspace H can be found as $u - pr_n(u) = u - \frac{u \cdot n}{||n||^2} p$.

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Problem 6. Find the matrix form of the linear transformation in $\mathbb{R}^2$ that reflects every vector about the line parallel to the vector $(1, 2)$ (show your work).

Solution: We did it in class (three times!).

Problem 7. Find the rank and nullity of this matrix

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{pmatrix}.$$

Answer: Reduce to the RREF, rank is the number of pivots, 3, the nullity is the number of columns minus the number of pivots, $4 - 3 = 1$.

Problem 8. (Bonus problem, extra credit) Show that if A and B are $m \times n$-matrices of the same rank. Then there exist invertible matrices C, D such that $A = CBD$.

Solution: Since the ranks of A and B are the same, the matrices $RREF(A)$ and $RREF(B)$ have the same number of pivots. By using column transformations, we can reduce both $RREF(A)$ and $RREF(B)$ to the same matrix U. Therefore one can get from B to A (via U) by a sequence of row and column transformations. It remains to notice that row transformations are the same as multiplying by elementary matrices on the left, and column transformations are the same as multiplying by elementary matrices on the right, and recall that elementary matrices are invertible, and products of invertible matrices are invertible too.