MIDTERM TEST 2

MATH 204

In every problem, you should show your work, not just the final result. You should also explain the steps of your solution. Try to consider all possible cases.

Problem 1. Let A, B be square matrices of the same size. Suppose that $B \neq 0$ and $AB = 0$. Prove that $\det(A) = 0$.

Solution. Suppose that $\det(A) \neq 0$ and $AB = 0$, $B \neq 0$. Then by a theorem about invertible matrices, A is invertible. Multiplying the equality $AB = 0$ by A^{-1} on the left, we get $B = A^{-1}0 = 0$, a contradiction.

Problem 2. A square matrix A is called *orthogonal* if $A^{-1} = A^{tr}$. Show that if A is orthogonal, then $\det(A) = \pm 1$.

Solution. Since $\det(A^{-1} = \frac{1}{\det(A)}, \det(A^{tr}) = \det(A)$, we get (by taking the determinants of both sides) $\frac{1}{\det(A)} = \det(A)$. Hence $(\det(A))^2 = 1$, so $\det(A) \pm 1$.

Problem 3. Find the adjoint of the following matrix: $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$

Answer: $\begin{pmatrix} -1 & 2 & -2 \\ -1 & -5 & -2 \\ -3 & -1 & 1 \end{pmatrix}$.

Problem 4. Find the determinant $\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}$

Solution. This is a Vandermonde determinant. So it is equal to $(2-1)(3-1)(4-1)(3-2)(4-2)(4-3) = 12$.

Problem 5. Let $A = (1,1,1), B = (2,3,4), C = (3,3,4), D = (2,1,1)$ be four points in $\mathbb{R}^3$.

a) Verify that $ABCD$ is a parallelogram.

Solution. $\vec{AB} = (1,2,3), \vec{DC} = (1,2,3)$, so the two opposite sides are parallel and of the same length. Hence the quadrangle is a parallelogram.

b) Find the area of $ABCD$.

Solution. $\vec{AB} = (1,2,3), \vec{AC} = (1,0,0)$. The cross-product is $(0,3,-2)$. The magnitude of the cross-product, $\sqrt{13}$, is the area of the parallelogram.

Problem 6. a) Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors $\vec{a} = (1,2,3), \vec{b} = (1,0,1), \vec{c} = (1,2,0)$. b) Are these vectors parallel to the same plane?

Solution. The volume is equal to the absolute value of the mixed product: $\left| \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \right| = 6.$ Since the volume is not 0, the vectors are not parallel to the same plane.

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Problem 7. A line l in $\mathbb{R}^3$ passes through the point $(1,1,1)$ and is parallel to the vector $(2,1,2)$. Find the distance from the point $(1,0,0)$ to that line.

Solution. Let $P=(1,1,1), Q=(1,0,0)$. Then $\vec{P}Q=(0,-1,-1)$. The area of the parallelogram spanned by $\vec{P}Q$ and $(2,1,2)$ is the magnitude of the cross-product, $\|(1,2,-2)\|=\sqrt{9}=3$. The distance from Q to the line is the distance from Q to the opposite side of the parallelogram, that is $\frac{3}{\|(2,1,2)\|}=\frac{3}{3}=1$.

Problem 8. (Bonus problem, extra credit) Show that

$$\det\begin{pmatrix} 1+x_1 & x_2 & x_3 & ... & x_n \\ x_1 & 1+x_2 & x_3 & ... & x_n \\ x_1 & x_2 & 1+x_3 & ... & x_n \\ ... & ... & ... & ... & ... \\ x_1 & x_2 & x_3 & ... & 1+x_n \end{pmatrix} = 1+x_1+x_2+...+x_n.$$

Solution. The determinant does not change if we add all the columns $\#2,3,...,n$ to the first column. The resulting matrix will be then equal to

$$\begin{pmatrix} 1+x_1+x_2+...+x_n & x_2 & x_3 & ... & x_n \\ 1+x_1+x_2+...+x_n & 1+x_2 & x_3 & ... & x_n \\ 1+x_1+x_2+...+x_n & x_2 & 1+x_3 & ... & x_n \\ ... & ... & ... & ... & ... \\ 1+x_1+x_2+...+x_n & x_2 & x_3 & ... & 1+x_n \end{pmatrix}.$$

The determinant of that matrix is equal to $(1+x_1+...+x_n)$ times the determinant of this matrix (if a column of a matrix is multiplied by a number, the determinant gets multiplied by the same number):

$$\begin{pmatrix} 1 & x_2 & x_3 & ... & x_n \\ 1 & 1+x_2 & x_3 & ... & x_n \\ 1 & x_2 & 1+x_3 & ... & x_n \\ ... & ... & ... & ... & ... \\ 1 & x_2 & x_3 & ... & 1+x_n \end{pmatrix}$$

Now subtract the first column of that matrix (multiplied by $x_2, x_3, ..., x_n$) from the other columns. The determinant does not change but the matrix turns into

$$\begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 1 & 1 & 0 & ... & 0 \\ 1 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 1 & 0 & 0 & ... & 1 \end{pmatrix}$$

which is a triangular matrix with determinant 1. Hence the determinant of the original matrix is $1+x_1+...+x_n$.