MIDTERM TEST 1

MATH 204

In every problem, you should show your work, not just the final result. You should also explain the steps of your solution. Try to consider all possible cases. This is an open book test, but you may not use help of other humans. The test is due at the beginning of the class on Thursday, January 31. You can use any part of the $46\frac{3}{4}$ hours between classes.

Problem 1. For which values of a the following system has unique solution? Check your answer.

$$\begin{cases} 2x+3y+az &= 5 \\ 3x-y+z &= -2 \\ 5x-y-z &= -3 \\ 10x+y+15z &= 0 \end{cases}$$

Answer: $a = 15$. **Solution.** The augmented matrix of the system is:

$$\begin{pmatrix} 2 & 3 & a & 5 \\ 3 & -1 & 1 & -2 \\ 5 & -1 & -1 & -3 \\ 10 & 1 & 15 & 0 \end{pmatrix}.$$

Reducing it to the RREF, we obtain after a few first steps:

$$\begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2}a & \frac{5}{2} \\ 0 & 1 & -\frac{2}{11}+\frac{3}{11}a & \frac{19}{11} \\ 0 & 0 & -\frac{28}{11}-\frac{2}{11}a & -\frac{9}{11} \\ 0 & 0 & \frac{137}{11}-\frac{13}{11}a & -\frac{9}{11} \end{pmatrix}.$$

Case 1. If $a = -14$, then the $(3,3)$-entry of this matrix is 0 and we cannot divide the third row by it. But if we plug in $a = -14$, the third row will become $(0,0,0,-\frac{9}{11})$, so the system does not have a solution. Hence we can assume that $a \neq -14$.

Case 2. Assuming $a \neq -14$, we can divide the third row by $-\frac{28}{11} - \frac{2}{11}a$ and continue the process of reducing the matrix. After one more step, we get the following matrix:

$$\begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2}a & \frac{5}{2} \\ 0 & 1 & -\frac{2}{11}+\frac{3}{11}a & \frac{19}{11} \\ 0 & 0 & 1 & \frac{9/2}{(a+14)} \\ 0 & 0 & 0 & \frac{9}{2}\left(\frac{-\frac{137}{11}+\frac{13}{11}a}{a+14}\right)-\frac{9}{11} \end{pmatrix}.$$

If the $(4,4)$-entry of that matrix is not 0, then the system does not have a solution. So the only possibility for the system to have unique solution can occur when $9/2\left(-\frac{137}{11}+\frac{13}{11}a\right)(a+14)^{-1} - \frac{9}{11} = 0$. Solving for a, we get $a = 15$. So a must be equal 15. Plugging $a = 15$ in the original system of equations, and reducing to RREF, we get the unique solution: $x = -\frac{10}{29}, y = \frac{65}{58}, z = \frac{9}{58}$.

Problem 2. Let A, C be the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -2 \end{pmatrix}, C = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}.$$

Find all matrices B such that $AB = C$. Check your answer.

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Solution. The matrix B must have dimensions $(3,2)$. So $B = \begin{pmatrix} x & u \\ y & v \\ z & w \end{pmatrix}$ and we need to find x, y, z, u, v, w. In order to solve the matrix equation $AB = C$, we need to consider the augmented matrix $(A \mid C)$ and reduce it to the RREF. The result is the following:

$$\begin{pmatrix} 1 & 0 & 1 & -3 & -4 \\ 0 & 1 & 1 & 3 & 4 \end{pmatrix}$$

Hence the general solution is $x = -3 - z, y = 3 - z, u = -4 - w, v = 4 - w$. Hence B has the form $B = \begin{pmatrix} -3-z & -4-w \\ 3-z & 4-w \\ z & w \end{pmatrix}$.

In order to **check**, we need to multiply AB, the result is C.

Answer:$B = \begin{pmatrix} -3-z & -4-w \\ 3-z & 4-w \\ z & w \end{pmatrix}$.

Problem 3. Suppose that the economy consists of three sectors: Coal, Electric Power, Steel. Each year, Coal sells 40% of its output to Electric Power, and 60% to Steel; Electric power sells 60% to Coal, 20% to Steel and 20% to itself; Steel sells 40% to Coal, 50% to Electric Power, and 10% to itself. What should be the prices for the annual outputs of these sectors so that each sectors's income coincides with its expenses?

Solution. Let p_c be price of the Coal output, p_e be the price of the Electric Power, and p_s be the price of Steel. Then p_c must be equal to the total expenses of the Coal sector. The expenses are the price of Electric Power and the price of Steel used by the Coal sector. Hence we have the equation

$$p_c = .6p_e + .4p_s.$$

Similarly for the other two sectors:

$$p_e = .4p_c + .2p_e + .5p_s,$$
$$p_s = .6p_c + .2p_e + .1p_s.$$

Hence we get the following system of equations:

$$\begin{cases} -p_c + .6p_e + .4p_s & = 0 \\ .4p_c - .8p_e + .5p_s & = 0 \\ .6p_c + .2p_e - .9p_s & = 0 \end{cases}$$

The general solution is $p_c = \frac{31}{28}p_s, p_e = \frac{33}{28}p_s$.

Answer: The price for the Coal sector output should be $\frac{31}{28}$ of the price for the Steel sector output, the price of the Electric power sector should be $\frac{33}{28}$ of the price for the Steel sector.

Problem 4. For every $n = 2, 3, 4, \ldots$, find a square $n \times n$-matrix A such that $A^n = O_n$ but $A^{n-1} \neq O_n$. Explain your answer (you may need to use mathematical induction in order to justify your answer).

Solution. There are many examples. For instance, let A be the $n \times n$-matrix with $a_{1,2} = a_{2,3} = \ldots = a_{n-1,n} = 1$ and all other entries equal 0. Then $A^n = O_n$ but A^{n-1} is the matrix with $(1, n)$-entry 1 and all other entries 0. **Explanation.** In order to prove our statement, we need to multiply A by itself n times.

By induction, let us prove that *for every $i = 1, 2, \ldots, n-1$, A^i is the matrix where $a_{1,i+1} = a_{2,i+2} = \ldots = a_{n-i,n} = 1$ and all other entries are equal to* 0.

The statement is true for $i = 1$ by the definition of A. Suppose it is true for some i. We need to prove it for $i + 1$. But $A^{i+1} = AA^i$. Multiplying these two matrices, we get the result: the row number j in A has 1 in the $j + 1$-st place and all other places have 0; the k-th column of A^i has 1 in the $k - i$-th place and all other entries 0; the product of row number j of A by the column number k in A^i gives 1, provided the 1's match, and gives 0 otherwise. The 1's match only if $j = 1, k = i + 2$ or $j = 2, k = i + 3$, and so on. This gives the formula for A^{i+1}: the only non-zero entries of this matrix are $a_{1,i+2}, .., a_{n-i-1,n}$, these entries are equal to 1.

This proves our statement for all $i = 1, ..., n-1$. In particular, A^{n-1} has exactly one non-zero entry, the $(1, n)$-entry. So $A^{n-1} \neq O_n$. But multiplying AA^{n-1}, we get $A^n = O_n$.

Problem 5. Is this matrix invertible? If yes, calculate the inverse and check the answer $\begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 0 & 4 \\ 2 & 1 & -1 & 3 \\ -1 & 0 & 5 & 7 \end{pmatrix}$.

Solution. Consider the augmented matrix $(A \mid I_4)$ where A is the given matrix. Reducing to the rref, we get

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{7} & 0 & -\frac{5}{14} & \frac{5}{14} & \frac{1}{14} \\ 0 & 1 & 0 & \frac{29}{7} & 0 & \frac{9}{14} & \frac{5}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{10}{7} & 0 & -\frac{1}{14} & \frac{1}{14} & \frac{3}{14} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

The first 4 columns of this matrix do not form I_4. So A is *not* invertible.