

## FINAL TEST

MATH 204

In every problem, you should show your work, not just the final result. You should also explain the steps of your solution. Try to consider all possible cases.

**Problem 1** (10 points). A square matrix  $A$  is called *orthogonal* if  $A^{-1} = A^{tr}$ . Suppose that  $T$  is a linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  that maps the standard basis to an orthonormal basis (that is  $\{T(e_1), \dots, T(e_n)\}$  is an orthonormal basis of  $\mathbb{R}^n$  where  $\{e_1, \dots, e_n\}$  is the standard basis). Prove that the matrix of  $T$  in the standard basis is orthogonal.

**Solution:** The columns of matrix  $A$  of  $T$  in the standard basis are the vectors  $T(e_1), \dots, T(e_n)$ . These are also the rows of the transpose  $A^{tr}$ . The entries of the matrix  $A^{tr}A$ , are dot products of the rows of  $A^{tr}$  by the columns of  $A$ . Since  $T(e_1), \dots, T(e_n)$  form an orthonormal basis, the dot product of every vector  $T(e_i)$  with itself is 1, and all the other dot products  $T(e_i) \cdot T(e_j)$  are equal to 0. Hence  $AA^{tr} = I$ , that is  $A^{-1} = A^{tr}$ , so  $A$  is an orthogonal matrix.

**Problem 2** (10 points). Find all values of number  $a$  for which this system of equations has a non-trivial solution:

$$\begin{cases} 2x + ay + z = 0 \\ 3x - ay + z = 0 \\ 5x - ay + 3az = 0 \end{cases}$$

**Solution:** For this system to have non-trivial solution, the determinant of the coefficient matrix must be 0. The determinant is  $-15a^2 + 9a$ . **Answer:**  $a = 0$  or  $a = 3/5$ .

**Problem 3** (Each problem is 10 points worth). a) Find an orthonormal basis of the solution space  $H$  of the system of equations

$$\begin{cases} 2x - 2y + z - 2w = 0 \\ 3x - y + z - w = 0 \end{cases}$$

(in  $\mathbb{R}^4$ ).

**Solution:** A basis of the solution space  $H$  is  $\{(-1, 1, 4, 0), (-1, 0, 4, 1)\}$ . Applying the Gram-Schmidt, we get an orthonormal basis  $e_1 = \frac{1}{3\sqrt{2}}(-1, 1, 4, 0)$ ,  $e_2 = \frac{1}{\sqrt{70}}(-1/3, -\frac{17}{3}, 4/3, 6)$ .

b) Find the matrix (in the standard basis) of the linear transformation  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$  that projects every vector in  $\mathbb{R}^4$  onto  $H$ .

**Solution:** The transformation takes every vector  $v$  to the projection vector  $(v \cdot e_1)e_1 + (v \cdot e_2)e_2$ . Applying this formula to the basic vectors of the standard basis  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ ,  $(0, 0, 0, 1)$ , we obtain the four columns of the matrix of the transformation:

$$\begin{pmatrix} \frac{2}{35} & -\frac{1}{35} & -\frac{8}{35} & -\frac{1}{35} \\ -\frac{1}{35} & \frac{18}{35} & \frac{4}{35} & -\frac{17}{35} \\ -\frac{8}{35} & \frac{4}{35} & \frac{32}{35} & \frac{4}{35} \\ -\frac{1}{35} & -\frac{17}{35} & \frac{4}{35} & \frac{18}{35} \end{pmatrix}.$$

**Problem 4** (10 points). Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation whose matrix  $A$  in the standard basis is upper triangular, and all diagonal entries of  $A$  are different. Show that there exists a basis of  $\mathbb{R}^n$  where the matrix of  $T$  is diagonal.

**Solution:** Let  $a_1, \dots, a_n$  be the diagonal entries of the matrix  $A$ . Then the characteristic polynomial  $\det(A - \lambda I)$  is  $(a_1 - \lambda) \dots (a_n - \lambda)$ . The eigenvalues are  $a_1, \dots, a_n$ . Since these numbers are all different by the assumption, each eigenvalue has algebraic multiplicity 1. The geometric multiplicity must be also equal to 1 (it cannot be 0 and cannot exceed the algebraic multiplicity). Hence  $T$  has a basis of eigenvectors. In that basis, the matrix of  $T$  is diagonal with  $a_1, \dots, a_n$  on the diagonal.

**Problem 5** (10 points). What are the rank and the nullity of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 2 \\ 3 & 3 & 5 & 6 \\ 2 & 2 & 1 & 1 \end{pmatrix} ?$$

**Answer:** 3 and 1.

**Problem 6** (10 points). Find a basis for the span of the following vectors in  $\mathbb{R}^4$ :

$$(1, 2, 0, -1), (1, -1, -2, 3), (2, 1, -2, 2), (-1, 1, -2, 5).$$

**Answer:**  $(1, 2, 0, -1), (1, -1, -2, 3), (-1, 1, -2, 5)$ .

**Problem 7** (10 points). A plane  $\Pi$  in  $\mathbb{R}^3$  passes through the point  $P = (1, 1, 1)$  and is perpendicular to the vector  $(2, 1, 2)$ . Find the distance from the point  $Q = (1, 0, 0)$  to that plane.

**Solution:** The distance is the length of the projection of the vector  $QP = (0, 1, 1)$  onto the normal vector of the plane,  $(2, 1, 2)$ . **Answer:** 1.

**Problem 8** (10 points). Suppose that a square matrix  $A$  can be reduced to the identity matrix by the following row transformations:  $[R_1 \rightarrow R_1 - 2R_2], [R_2 \rightarrow \frac{1}{9}R_2], [R_3 \leftrightarrow R_5], [4R_5]$ . Find the determinant of  $A$ .

**Answer:**  $-\frac{9}{4}$ : the first transformation does not change the determinant, the second transformation divides it by 9, the third transformation changes the sign, the fourth transformation multiplies it by 4; the resulting determinant is 1.

**Problem 9** (10 points). Suppose that a linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  has matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

in the standard basis. Find the eigenvalues of  $T$ . Find out if  $\mathbb{R}^3$  has a basis of eigenvectors of  $T$ , and if so, find such a basis and the matrix of  $T$  in that basis.

**Solution:** The eigenvalues are  $-1$  (multiplicity 2) and  $2$ . The  $1$ -eigenspace is spanned by  $(-1, 1, 0), (-1, 0, 1)$ , the  $2$ -eigenspace is spanned by  $(1, 1, 1)$ . These three vectors form a basis of eigenvectors of the transformation. The matrix of the transformation in this basis is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Problem 10.** (Bonus problem, extra credit) Let us define the dot product on the space of polynomials of degree 2,  $P_2$ , as  $f(x) \cdot g(x) = \int_0^1 f(x)g(x)dx$ . Find an orthonormal basis of  $P_2$ .

**Solution:** We need to apply the Gram-Schmidt to the standard basis  $1, x, x^2$ . Let  $a_1 = 1$ . Since the norm of 1, that is  $\int_0^1 1^2 dx$ , is 1,  $a_1$  is a unit vector, so  $e_1 = 1$ . Let  $a_2 = x$ , then  $b_2 = a_2 - (a_2 \cdot e_1)e_1 = x - \int_0^1 x dx = x - \frac{1}{2}$ . The norm of  $x - 1/2$  is  $1/\sqrt{12}$ . So  $e_2 = 2\sqrt{3}x - \sqrt{3}$ . Then  $a_3 = x^2$ ,  $b_3 = a_3 - (a_3 \cdot e_1)e_1 - (a_3 \cdot e_2)e_2 = x^2 - x + 1/6$ . The norm of  $b_3$  is  $\frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}}$ , so  $e_3 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$ .