MATH 194
MIDTERM TEST 3.
NOVEMBER 20, 2014. SOLUTIONS.

Problem 1. Let $\vec{x} = (1, 1, -1, 1)^T$, $\vec{y} = (1, 2, 3, 4)^T$. Determine the distance and the angle between $\vec{x}$ and $\vec{y}$.

Solution: The distance is the length of $\vec{x} - \vec{y} = (0, -1, -4, -3)$, which is $\sqrt{26}$. The angle α is given by

$$\cos(\alpha) = \frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{x}|| ||\vec{y}||} = \frac{4}{2 \cdot \sqrt{30}} = \frac{2}{\sqrt{30}}.$$

Problem 2. Consider the data:

x	-1	0	1	2
y	2	3	4	1

. Find the best least squares fit to this data by a linear polynomial.

Solution: We are looking for the least squares fit of the form $ax + b$, i.e., the best "non-solution" of the system of equations

$$\begin{cases} a * (-1) + b = 2 \\ a * 0 + b = 3 \\ a * 1 + b = 4 \\ a * 2 + b = 1 \end{cases}.$$

The matrix A of this system is $\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$, and the vector of the right hand sides is $r = (2, 3, 4, 1)^T$. The least squares solution $u = (a, b)^T$ is obtained by solving the system $A^T A u = A^T r$. Solving it, we obtain $u = (-\frac{1}{5}, \frac{13}{5})$. Hence the answer is $-\frac{1}{5}x + \frac{13}{5}$.

Problem 3. Consider the vector space $C[-1, 1]$ with inner product defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Find an orthonormal basis for the subspace spanned by $1, x, x^2$.

Solution: We need to obtain three pairwise orthogonal functions e_1, e_2, e_3 of unit norm. Let $g_1 = 1$. The norm is $\sqrt{\int_{-1}^{1} g_1^2 dx} = \sqrt{2}$, so $e_1 = \frac{1}{\sqrt{2}}$. Let $g_2 = x$. Apply Gram-Schmidt to e_1, g_2, we get $n := g_2 - \langle g_2, e_1 \rangle e_1 = g_2$. Its norm is $\sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{2}{3}}$. Hence $e_2 = \sqrt{\frac{3}{2}}x$. Now apply Gram-Schmidt to $e_1, e_2, g_3 = x^2$. We get $n = g_3 - \langle g_3, e_1 \rangle e_1 - \langle g_3, e_2 \rangle e_2 = \left(x^2 - \frac{1}{3}\right)$. Its norm is $\sqrt{\frac{8}{45}}$. So $e_3 = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right)$.

Problem 4. Let $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$. Factor A into a product XDX^{-1} where D is a diagonal matrix.

Solution: The eigenvalues of matrix A are $2, 1$. The diagonal matrix then is $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Corresponding eigenvectors are $(-2, 1)^T$ and $(-\frac{3}{2}, 1)^T$. The transition matrix is $C = \begin{pmatrix} -2 & -\frac{3}{2} \\ 1 & 1 \end{pmatrix}$. Therefore $A = CDC^{-1}$.

Problem 5. Let $\vec{z}_1 = \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix}, \vec{z}_2 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$. Show that $\vec{z}_1, \vec{z}_2$ is an orthogonal basis of $\mathbb{C}^2$ and write the vector $\vec{z} = \begin{pmatrix} 2+4i \\ -2i \end{pmatrix}$ as a linear combination of $\vec{z}_1, \vec{z}_2$.

Solution. The dot product of $\vec{z}_1$ and $\vec{z}_2$ is 0 and the lengths of these vectors are equal to 1. Since $\mathbb{C}^2$ is 2-dimensional, these two vectors form an orthonormal basis of $\mathbb{C}^2$. The coordinates of $\vec{z}$ in that basis are $\langle \vec{z}, \vec{z}_1 \rangle = (2-4i)(1+i)/2 + (1-i)/2 \cdot 2i = 4$ and $\langle \vec{z}, \vec{z}_2 \rangle = 2\sqrt{2}$.

Problem 6. Show that for every square matrix A with real entries matrices AA^T and A^TA are conjugate.

Solution: Let us use the singular value decomposition $A = UDV$ where U, V are orthogonal, D is diagonal. Then $AA^T = UDVV^TD^TU^T = UD^2U^T = UD^2U^{-1}$ since U and V are orthogonal matrices, whence $U^T = U^{-1}, V^T = V^{-1}$. Similarly $A^TA = V^{-1}D^2V$. Therefore $A^TA = V^{-1}D^2V = V^{-1}U^{-1}UD^2U^{-1}V = (U^{-1}V)^{-1}AA^T(U^{-1}V)$, thus AA^T is conjugate to A^TA.

Problem 7. a) Find a singular value decomposition of the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) What is the best rank 1 approximation of this matrix?

Solution: a) The eigenvalues of AA^T are $2, 1$. So the singular values are $\sqrt{2}, 1$ (in that order since $\sqrt{2} > 1$). So the matrix E in the singular value decomposition $A = UEV$ is $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. The orthonormal basis of eigenvectors is $(1,0)^T, (0,1)^T$. So the matrix V is the identity matrix I. An orthonormal basis of eigenvectors of the matrix $AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ consists of vectors $\frac{1}{\sqrt{2}}(1,1,0)^T$, $(0,0,1)^T$ and $\frac{1}{\sqrt{2}}(-1,1,0)$ (belonging to the eigenvalues $2, 1, 0$). So the matrix U is $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$.

b) To get the best rank 1 approximation, we replace the lowest singular value (=1) by 0. The answer is $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$.