MATH 194
MIDTERM TEST 2.
SOLUTIONS

**Problem 1.** Let $\vec{v}_1 = (1, 2, 3)^T, \vec{v}_2 = (1, 1, -1)^T, \vec{v}_3 = (0, -1, -2)^T$.

(a) Show that these vectors form a basis in $\mathbb{R}^3$,

**Solution.** The determinant of the matrix $A$ whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is not zero, so the columns are linearly independent. Since the dimension of $\mathbb{R}^3$ is 3, these vectors form a basis of $\mathbb{R}^3$.

(b) Find the transition matrix from the standard basis to that basis,

**Solution.** Let $\mathcal{B}$ be that basis. Then the matrix $A$ is the transition matrix $[\mathcal{B} \to \mathcal{S}]$. The transition matrix $[\mathcal{S} \to \mathcal{B}]$ is then $A^{-1}$.

(c) Use the transition matrix to find coordinates of the vector $(2, 3, 1)^T$ in that basis.

**Solution.** The answer: $A^{-1}(2, 3, 1)^T$.

**Problem 2.** How many solutions will the linear system $Ax = b$ have

(a) if $b$ is in the column space of $A$ and the columns of $A$ are linearly independent?

(b) if $b$ is not in the column space of $A$

Explain your answer.

**Solution.**

(a) Since $b$ is in the column space of $A$, it is a linear combination of columns of $A$, hence there is a solution of the system $Ax = b$. Since the columns of $A$ are linearly independent in the reduced row echelon form of $A$ every column will have a pivot. Therefore the system $Ax = b$ does not have free unknowns, hence it has exactly one solution.

(b) Since $b$ is not in the column space of $A$, it is not a linear combination of columns of $A$, hence $Ax = b$ has no solutions.

**Problem 3.** Find the matrix of the projection of $\mathbb{R}^3$ onto the the plane $x - y - z = 0$.

**Solution.** A normal vector of the plane is $\vec{n} = (1, -1, -1)^T$. The vectors $\vec{a} = (1, 1, 0)^T$, $\vec{b} = (1, 0, 1)^T$ are parallel to the plane. These three vectors $\vec{n}, \vec{a}, \vec{b}$ form a basis $\mathcal{B}$ of $\mathbb{R}^3$. Let $[\mathcal{B} \to \mathcal{S}]$ be the corresponding transition matrix. In the basis $\mathcal{B}$, the matrix of the projection is $P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (because the projection of $\vec{n}$ is $\vec{0}$, the projection of $\vec{a}$ is $\vec{a}$, the projection of $\vec{b}$ is $\vec{b}$. Then the matrix of the projection is $[\mathcal{B} \to \mathcal{S}]P[\mathcal{B} \to \mathcal{S}]^{-1}$.

**Problem 4.** (a) Is it possible for a matrix to have the vector $(1, 2, 3)$ in its row space and the vector $(2, -1, -1)^T$ in its null space?

**Solution.** No, it is not possible. If a vector $\vec{a}$ in the row space of a matrix, then it is a linear combination $\alpha_1\vec{r}_1 + ... + \alpha_n\vec{r}_n$ of the rows of the matrix. If $\vec{b}^T$ is in the null space of the matrix, then $\vec{r}_i\vec{b}^T = 0$ for each $i$. Therefore $\vec{a}\vec{b}^T$ should be equal to 0, but $(1, 2, 3)(2, -1, -1)^T = -3 \neq 0$.

(b) Give an example of a matrix having $(1, 2, 3)$ in its row space and $(-2, 1, 0)^T$ in its null space.

**Solution.** Let $A$ be the $1 \times 3$ matrix consisting of one row $(1, 2, 3)$. Then $A(-2, 1, 0)^T = 0$, hence $(-2, 1, 0)^T$ is in the null space of $A$. Clearly, $(1, 2, 3)$ is in the row space of $A$.

**Problem 5.** Find the matrix of the linear transformation of $\mathbb{R}^2$ which is the composition of dilation by $1/2$, rotation through 60 degrees and reflection about the line $y = x$.

**Solution.** The basic vectors $(1,0)$, and $(0,1)$ are mapped by this linear transformation as follows (we first apply the dilation, then the rotation, then the reflection):

$$(1,0)^T \to (\tfrac{1}{2},0)^T \to \begin{pmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$
$$\to \begin{pmatrix} \frac{\sqrt{3}}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$(0,1)^T \to (0,\frac{1}{2})^T \to \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{4} \\ \frac{1}{4} \end{pmatrix} \to \begin{pmatrix} \frac{1}{4} \\ -\frac{\sqrt{3}}{4} \end{pmatrix}$$

Therefore the matrix of this linear transformation is

$$\begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{3}}{4} \end{pmatrix}.$$

**Problem 6.** Prove that if $A, B, C$ are square matrices of the same size, and $A$ is similar to $B$, $B$ is similar to $C$, then $A$ is similar to $C$.

**Solution.** Since $A$ is similar to $B$, there exists a matrix $S$ such that $A = S^{-1}BS$. Since $B$ is similar to $C$, there exists a matrix $T$ such that $B = T^{-1}CT$. Therefore $A = S^{-1}T^{-1}CTS = (TS)^{-1}C(TS)$. Thus $A$ is similar to $C$.

**Problem 7.** Consider functions $1+x^2, x-1, x^2+x+1$ as elements of the vector space $C[0,1]$. Are these functions linearly independent? Explain your answer.

**Solution.** The Wronskian of these functions is the determinant of the matrix

$$\begin{pmatrix} 1+x^2 & x-1 & x^2+x+1 \\ 2x & 1 & 2x+1 \\ 2 & 0 & 2 \end{pmatrix}$$

which is equal to $-2$. Since it is not equal to 0, the functions are linearly independent.