Math 194
Midterm Test 1.

Solutions

Problem 1. Consider a linear system of equations

$$\begin{cases} x+y+z=2 \\ x+2y+3z=3 \\ x+3y+az=b \end{cases}$$

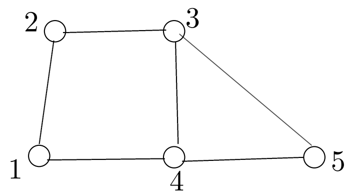
(a) For which values a and b will this system have infinitely many solutions?
(b) For which values of a and b will this system be inconsistent?

Solution. The augmented matrix can be reduced to the matrix where the last row is $(0,0,a-5|b-4)$ (the other rows have pivots not in the last column). Thus if $a \neq 5$, all variables are pivotal, hence one solution, if $a=5, b=4$, then the system has an independent unknown and no pivots in the last column, hence infinitely many solutions, and if $a=5, b\neq 4$, there are no solutions because there will be a pivot in the last column.

Problem 2. Let A be a non-zero symmetric 2×2-matrix. Is it possible that A^2 is the zero matrix? Justify your answer.

Solution. No. A symmetric matrix has this form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. If its square is 0, then $a^2+b^2=0$ and $b^2+c^2=0$, hence $a=b=c=0$ and the matrix is zero.

Problem 3. Consider the following graph:



How many walks of length 3 are from vertex 1 to vertex 4? Justify your answer.

Solution. Consider the $(1,4)$-entry of the 3-d power of the incidence matrix. The answer is 5.

Problem 4. Express the matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

as a product of elementary matrices.

Solution. Reduce the matrix to RREF. Let E_1, E_2, E_3 be elementary matrices corresponding to the row operations used in this reduction. Then the matrix is equal to $E_1^{-1}E_2^{-1}E_3^{-1}$. One possible answer: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Problem 5. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution. The inverse is $\begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$.

Problem 6. Let A be an 4 by 4 matrix with entries a_{ij}. In the expression of the determinant of A what is the sign of the product $a_{13}a_{22}a_{34}a_{41}$?

Solution. The corresponding permutation is $(3, 2, 4, 1)$. One needs two switches to get $(1, 2, 3, 4)$ from it: switch the first and the fourth numbers, then switch the third and the fourth numbers. The number of switches is even, so the sign is $+$.

Problem 7. Let A be the n by n matrix with 1's on the main diagonal, 1's on the diagonal above the main diagonal, -1's on the diagonal just below the main diagonal, and 0's everywhere else. Show that the determinant of A is a Fibonacci number.

Solution. For $n = 1$, the determinant is 1, for $n = 2$ the determinant is 2. Note that 1 and 2 are the first two Fibonacci numbers. For an arbitraty $n \geq 3$ let F_n be the determinant. Expand along the first column, we get the sum of two determinants, of which the first one is F_{n-1}. The second determinant, after expanding along the first row, turns into F_{n-2}. Thus $F_n = F_{n-1} + F_{n-2}$ for every $n > 2$ which is the Fibonacci recurrence relation. Therefore F_n is a Fibonacci number for every n.