Curvature and rigidity: results of surgery

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Some recent results

Introduction
The curvature problem
The rigidity problem
Noncompact venue
Conclusion

Joint work with Shmuel Weinberger and Guoliang Yu.

Let $M = \Gamma \backslash G/K$ be a noncompact arithmetic manifold whose $\mathbb{Q}$-rank is at least 3.

Theorem (BW 1999)

The manifold M admits a metric of positive scalar curvature.

Theorem (CW 2008)

Then M has a finite-sheeted cover N whose topological proper structure $S_p^{Top}(N)$ set is nontrivial; i.e. the manifold M is virtually properly nonrigid. (In some particular cases strictly so [2012].)

Low-dimensional results

Theorem (CWY 2010)

The only noncompact contractible 3-manifold with positive scalar curvature is $\mathbb{R}^3$.

Theorem (CWY 2010)

The only noncompact oriented 3-manifolds with positive scalar curvature are connected sums of space forms and $S^2 \times S^1$.

Some high-dimensional results

Theorem (CWY 2012)

There are (contractible) noncompact manifolds with uncountably many positive scalar curvature components.

Theorem (CWY 2012)

There are (contractible) manifolds M with a positively curved exhaustion but which itself cannot carry a positive scalar curvature metric.

Scalar Curvature

Given a Riemannian metric g on a manifold M of dimension n, we can define a function $\kappa_g : M \to \mathbb{R}$ measuring the *scalar curvature* of the manifold M at each point.

Definition

If g is a Riemannian metric on M of dimension n, the *scalar curvature* is a smooth function $\kappa_g : M \to \mathbb{R}$ obtained from the curvature tensor by contracting twice.

$$\frac{\mathrm{vol}_g B_r(M,p)}{\mathrm{vol}_{g_s} B_r(\mathbb{R}^n,0)} = 1 - \frac{\kappa_g(p)}{6(n+2)}\, r^2 + \cdots .$$

Volumes of Balls

Let M be an n-dimensional manifold endowed with a Riemannian metric g, and let $p \in M$. Suppose that $\kappa(p) \neq 0$; i.e. M is not *flat at p*. Then there is an $\varepsilon > 0$ such that, for all $r \in (0, \varepsilon)$, one of the following is true:

1. $\mathrm{vol}_g B_r(M, p) < \mathrm{vol}_g B_r(\mathbb{R}^n, 0)$;
2. $\mathrm{vol}_g B_r(M, p) > \mathrm{vol}_g B_r(\mathbb{R}^n, 0)$.

In these cases, we say that M is (1) *positively curved*, (2) *negatively curved* at p.

Trichotomy Theorem

Theorem (Kazdan-Warner)

Let M^n be a closed differentiable manifold of dimension n. Then M belongs to exactly one of the following three classes:

1. *those admitting some Riemannian metric g for which $\kappa_g > 0$ (positive manifolds);*
2. *those admitting no Riemannian metric h with $\kappa_h > 0$, but admitting a metric g with $\kappa_g \equiv 0$;*
3. *those admitting no Riemannian metric h with $\kappa_h \geq 0$, but admitting a metric g with $\kappa_g < 0$.*

Surgery

Definition

Let N be a manifold of dimension n and suppose that there is an embedding of $S^k \times D^{n-k}$ in N. Let M be the manifold obtained by glueing the complement of

$$S^k \times D^{n-k} \subset N \quad \text{and} \quad D^{k+1} \times S^{n-k-1}$$

along their common boundary $S^k \times S^{n-k-1}$. We say that M is obtained from N by *k-surgery* (or surgery of codimension $n-k$).

A theorem of Gromov-Lawson and Schoen-Yau

Theorem (GL 1980, SY 1979)

Let N be a closed manifold with a positive scalar curvature metric, not necessarily connected, and let M be obtained from N by surgery of codimension ≥ 3. Then M has a positive scalar curvature metric.

Corollary

The connected sum $M_1 \# M_2$ of two n-dimensional manifolds is obtained from their disjoint union by a 0-surgery. If M_1 and M_2 are closed manifolds of dimension $n \geq 3$ with pscm, then the connected sum $M_1 \# M_2$ also admits a metric of positive scalar curvature.

Aspherical manifolds

Definition

A compact manifold of the form $K(\pi, 1)$ is called *aspherical*.

In dimension 2 the aspherical manifolds are precisely the ones which lack a positive scalar curvature metric.

The Borel conjecture (for curvature): Any compact aspherical manifold lacks a metric of positive scalar curvature.

Another theorem of Gromov-Lawson

Theorem

If M is a simply connected manifold of dimension $n \geq 5$ that does not admit a spin structure, then M admits a metric of positive scalar curvature.

Dirac operator methods

Theorem (Lichnerowicz)

If M is a closed spin manifold of dimension $4k$, endowed with the Atiyah-Singer Dirac bundle S. If D is the Dirac operator on this bundle and ∇ is the standard Levi-Cività connection, then

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4}.$$

Corollary

If $\kappa > 0$, then the index of D, given by

$$ind(D) \equiv \dim \ker(D) - \dim \operatorname{coker}(D),$$

must vanish. Note: The Atiyah-Singer index theorem says that $ind(D)$ is equal to the topological invariant $\widehat{A}(M)$.

A necessary condition

Theorem (Atiyah, Singer 1963)

If the closed spin manifold M^{4k} admits a metric with positive scalar curvature, then $\widehat{A}(M) = 0$ in $\mathbb{Z}$.

Theorem (Hitchin)

If the closed spin manifold M^n admits a metric with positive scalar curvature, then $\alpha(M) = 0$ in $KO^{-n}(*)$, where

$$KO^{-n}(*) = \begin{cases} \mathbb{Z}, & n \equiv 0 \bmod 4, \\ \mathbb{Z}_2, & n \equiv 1, 2 \bmod 8, \\ 0, & \textit{otherwise}. \end{cases}$$

Indicial Receptacles

For particular types of manifolds M, we can define an Dirac-like index that lies in the following groups.

Atiyah	$\mathbb{Z}$	1963
Hitchin	$KO^{-*}(pt)$	1974
Gromov	$KO_*(B\pi)$	1980
Rosenberg	$KO_*(C_r^*\pi)$	1986
Roe	$K_*(C^*(M))$	1995

General idea: If M can be endowed with a positive scalar curvature metric, then the index vanishes.

The Gromov-Lawson-Rosenberg Conjecture

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Conjecture: (Borel) If $M = K(\pi, 1)$ is a closed aspherical manifold, then it is not positive.

Conjecture: (GLR) Suppose that M is a connected closed spin manifold of dimension $n \geq 5$. Then M is positive iff a particular Dirac index $\widehat{\alpha}(M)$ vanishes in $KO_*(C_r^*\pi)$.

Counterexample: Take $\pi = \mathbb{Z}_3 \times \mathbb{Z}^4$. (Schick 1998)

Summary of results

Necessary vanishing condition to positive scalar curvature in compact manifolds of dimension ≥ 5.

	spin	nonspin
simply connected	Hitchin invariant in $KO^{-*}(pt)$	none
not simply connected	Dirac index in $K_*(C_r^*\pi)$ (False: $\mathbb{Z}^4 \times \mathbb{Z}_3$)	Universal class in $H_n(\underline{B}\pi)$ (False: $\mathbb{Z}^4 \times \mathbb{Z}_3$)

Definition of $S(V)$

Let Cat = Top (topological), PL (piecewise linear) or Diff (smooth).

Definition

Let V be a connected space, say a finite CW-complex. A *Cat manifold structure* on V is a homotopy equivalence $M \to V$, where M is a Cat manifold.

Definition

Let $S^{Cat}(V)$ be the set of equivalence classes of manifold structures on V. Then $S^{Cat}(V)$ is called the *Cat structure set of* V.

Results about the structure set

1. If V does not satisfy Poincaré duality, then $S^{Cat}(V) = \emptyset$.
2. Milnor discovers that $S^{Diff}(S^7)$ has 28 elements and that $S^{Diff}(S^n)$ forms a group.
3. The Poincaré conjecture states that $S^{Top}(S^n)$ and $S^{PL}(S^n)$ are trivial; i.e. the n-sphere is *topologically and PL rigid*.
4. The structure set $S^{Top}(\mathbb{RP}^{4k+1})$ is nontrivial and finite. The structure set $S^{Top}(\mathbb{RP}^{4k+3})$ is infinite. (BL 1973)
5. If $n = 4k + 3$ and $\pi_1(M^n)$ has torsion, then $S^{Top}(M^n)$ is infinite. (CW 2004)

More results about the structure set

1. The structure set $S^{Top}(S^k \times S^n)$ is trivial iff k and n are both odd $(k + n \neq 3)$.
2. There are topological manifolds M for which $S^{PL}(M) = \emptyset$ and PL manifolds N for which $S^{Diff}(N) = \emptyset$.
3. There are PL manifolds M for which $S^{PL}(M)$ is nontrivial but $S^{Top}(M)$ is trivial.

The set $S^{Cat}(V)$ measures the extent to which a homotopy equivalence $M \to V$ is homotopy equivalent to a homeomorphism.

Mostow's rigidity theorem

Theorem (Mostow 1973, Marden 1974, Prasad 1973)

Suppose M and N are complete finite-volume hyperbolic n-manifolds with $n \geq 3$. If there exists an isomorphism $f : \pi_1(M) \to \pi_1(N)$, then it is induced by a unique isometry from M to N.

Another version is to state that any homotopy equivalence from M to N can be homotoped to a unique isometry.

Borel conjecture (1953): If $M = K(\pi, 1)$ is compact, then any homotopy equivalence from M to N can be homotoped to homeomorphism; i.e. $S^{Top}(M)$ is trivial.

A brief review of the surgery exact sequence

Fundamental Result in Surgery (Browder, Novikov, Sullivan, Wall 1960s; Kirby, Siebenmann 1970s): If M is a Cat manifold of dimension $n \geq 5$ with fundamental group π, there is an exact sequence

$$\cdots \to L_{n+1}(\mathbb{Z}\pi) \to S^{Cat}(M) \to [M\colon F/Cat] \to L_n(\mathbb{Z}\pi)$$

where F/Cat is a particular classifying space encoding bundle data.

The Wall groups $L_n(\Gamma)$

If π is trivial, then

$$L_n(\mathbb{Z}\pi) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \bmod 4 \ (\textit{signature}), \\ 0 & \text{if } n \equiv 1 \bmod 4, \\ \mathbb{Z}_2 & \text{if } n \equiv 2 \bmod 4 \ (\textit{Arf invariant}), \\ 0 & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

We can use the Poincaré conjecture and the sequence

$$[\Sigma S^6 \colon F/Top] \to L_7(\mathbb{Z}) \to S^{Top}(\Sigma S^5) \to$$

$$[\Sigma S^5 \colon F/Top] \to L_6(\mathbb{Z}) \to S^{Top}(S^5) \to [S^5 \colon F/Top] \to L_5(\mathbb{Z})$$

to conclude that $\pi_n(F/Top) \cong L_n(\mathbb{Z})$ for large n.

Arithmetic manifolds

Let G be a connected, real, semisimple Lie group with finite center and Γ a lattice in G. Let K be a maximal compact subgroup of G.

Topic of interest: the locally symmetric spaces $\Gamma\backslash G/K$.

Oftentimes we include the additional assumptions:

- the center of G is trivial;
- the lattice Γ is torsion-free.

Rational rank

Definition

The *rational rank* $\operatorname{rank}_{\mathbb{Q}}^{G}(\Gamma)$ of Γ in G is the smallest r, for which there exists a collection of finitely many (closed, simply connected) r-dimensional flats, such that all of $\Gamma\backslash G/K$ is within a bounded distance of the union of these flats.

Example

1. If Γ is cocompact in G, then $\operatorname{rank}_{\mathbb{Q}}^{G}(\Gamma) = 0$.
2. $\operatorname{rank}_{\mathbb{Q}}^{SL(n,\mathbb{R})}(\mathrm{SL}(n,\mathbb{Z})) = n - 1$
3. $\operatorname{rank}_{\mathbb{Q}}^{SO(m,n)}(\mathrm{SO}(m,n)_{\mathbb{Z}}) = \min\{m,n\}$

More geometrically, the rational rank of Γ is the dimension of the tangent cone at infinity of $\Gamma\backslash G/K$. Also

$$\operatorname{rank}_{\mathbb{Q}}^{G}(\Gamma) + \operatorname{cd}(\Gamma\backslash G/K) = \dim(\Gamma\backslash G/K).$$

Locally symmetric spaces

The topological analogue of characteristic class obstructions to positive scalar curvature are similar obstructions to proper homotopy equivalence. By Farrell-Jones we get the ridigity versions:

Theorem (Farrell-Jones 1998)

Let $M = \Gamma \backslash G / K$ as previously described.

1. *If Γ is arithmetic of rational rank 0 or 1, then M is properly rigid.*
2. *If the rational rank is 2, then M is properly rigid if one knows that the Borel conjecture for the fundamental group at infinity is known.*

Some recent results

Can Borel's conjecture be extended to provide a homotopy from homotopy equivalence of a nonuniform lattice quotient to a homeomorphism?

Theorem (CW 2008)

Let $M = \Gamma \backslash G/K$ be an arithmetic manifold whose $\mathbb{Q}$-rank is at least 3. Then M has a finite-sheeted cover N whose topological proper structure $S_p^{Top}(N)$ set is nontrivial; i.e. the manifold M is virtually properly nonrigid.

Take $\Gamma' \leq \Gamma$ of finite index and form the cover N, so that $H^2(N, \mathbb{Z}_2) \neq 0$. Use Sullivan's result that $F/Top = Z \times \prod_{k=1}^{\infty} K(\mathbb{Z}_2, 4k-2)$ for some space Z. The proper structure group satisfies

$$S^p(N) = [N, F/Top] = [N, K(\mathbb{Z}_2, 2)] \times [N, K(\mathbb{Z}_2, 6)] \times \cdots \times [N, Z].$$

However $[N, K(\mathbb{Z}_2, 2)] = H^2(N, \mathbb{Z}_2)$.

Failure to extend Mostow

Theorem (CW 2008)

Under the same hypotheses as above, the space M has finite-sheeted covers N whose proper structure sets $S_p^{Top}(N)$ are arbitrarily large.

Corollary

Mostow's rigidity theorem cannot be weakened to provide a proper version of Borel's conjecture for manifolds of noncompact type.

Failure to extend Mostow

Theorem (CW 2012)

In the case of $G/K = SL_n(\mathbb{R})/SO_n(\mathbb{R})$*, one can choose a lattice* Γ *so that the proper structure set of* $\Gamma\backslash G/K$ *contains elements that are not merely self-homotopy equivalences.*

Results in curvature

Let $M = \Gamma \backslash G / K$ be an arithmetic manifold.

1. Gromov and Lawson (1983): If $\text{rank}_{\mathbb{Q}}^{G}(\Gamma) \leq 1$, then M admits no metric of positive scalar curvature.
2. Block and Weinberger (1999): If $\text{rank}_{\mathbb{Q}}^{G}(\Gamma) \geq 3$, the manifold M admits a metric of positive scalar curvature.
   1. If $q = 0$, then M is compact.
   2. If $q = 1$, then $\pi_1^{\infty}(M) \to \pi_1(M)$ is injective.
   3. If $q = 2$, there is an exact sequence

   $$1 \to \mathbb{F}_{\infty} \to \pi_1^{\infty}(M) \to \pi_1(M) \to 1.$$

   4. If $q \geq 3$, then $\pi_1^{\infty}(M) = \pi_1(M)$.

Summary of known results

Let $M = \Gamma \backslash G / K$ be an arithmetic manifold. Recall that we take these spaces as a way to generalize a particular subclass of aspherical closed manifolds to aspherical manifolds of noncompact type.

$\mathbb{Q}$-rank	does pscm exist?	is (properly) rigid?
0	No GL	Yes FJ
1	No GL	Yes FJ
2	No BW	Yes FJ, BL
≥ 3	Yes BW	(No) CW

Low-dimensional results

Theorem (CWY 2010)

The only noncompact contractible 3-manifold with positive scalar curvature is $\mathbb{R}^3$.

Theorem (CWY 2010)

The only noncompact oriented 3-manifolds with positive scalar curvature are connected sums of space forms and $S^2 \times S^1$.

Suppose that M is an oriented n-manifold with $\Gamma = \pi_1(M)$ and Σ is a compact separating codimension 1 hypersurface partitioning M into M_0 and M_1. Denote by Σ_Γ the Γ-lift of Σ. Assume that the strong Novikov conjecture holds for Γ and that the image of $[D_\Sigma]$ is nonzero under the map $f_* : K^\Gamma_{*-1}(\widetilde{\Sigma}) \to K_*(B\Gamma)$. Then $\mathrm{ind}(\widetilde{D})$ is nonzero in $K_*(C^*_{\Gamma,b}(\widetilde{M}))$.

High-dimensional results

Theorem (CWY 2012)

There are (contractible) noncompact manifolds with uncountably many positive scalar curvature components.

Triangulate the boundary of the Davis manifold and reflect across the triangulation.

Exhaustions

Theorem (CWY 2012)

There are (contractible) manifolds M with a positively curved exhaustion but which itself cannot carry a positive scalar curvature metric.

Use the first derived functor $\varprojlim^1$ to find exhaustions that have incompatible positive scalar curvature metrics.