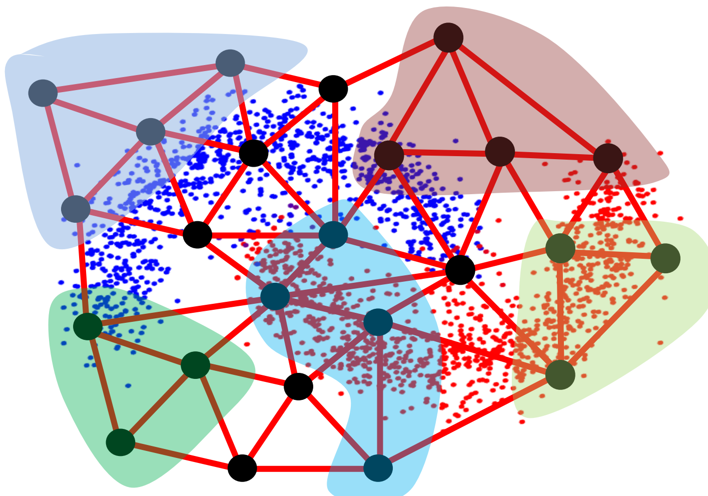
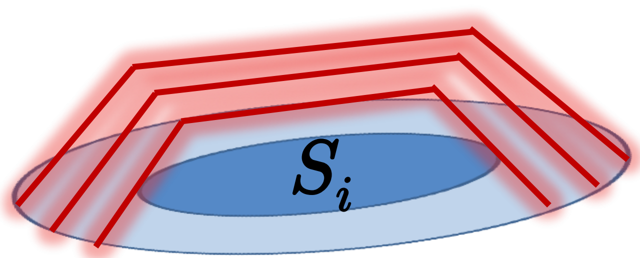
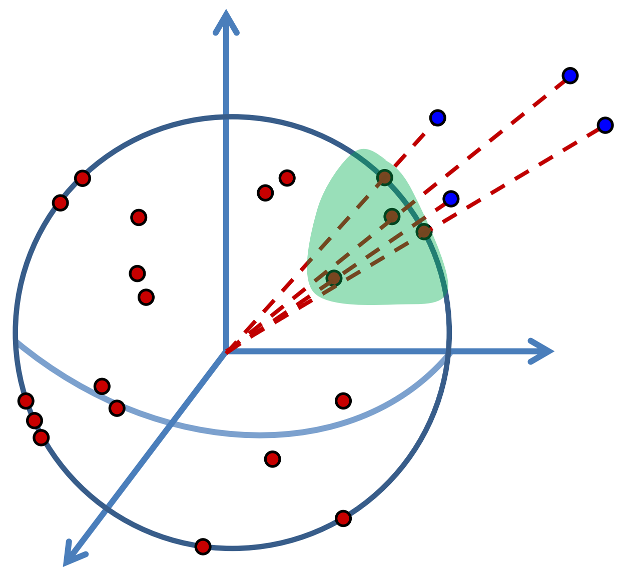
multi-way spectral partitioning and higher-order Cheeger inequalities

James R. Lee
University of Washington

Shayan Oveis Gharan
Luca Trevisan
Stanford University







spectral theory of the Laplacian

Consider a d-regular graph $G = (V, E)$.

Define the **normalized Laplacian**: $L = I - \frac{1}{d} A$

(where A is the adjacency matrix of G)

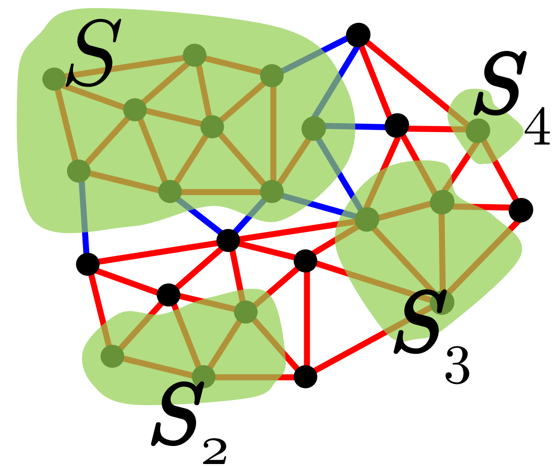
L is positive semi-definite with spectrum

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|} \leq 2$$

Fact: $\lambda_2 = 0 \iff G$ is disconnected.

$$\lambda_2 = \min_{f \perp \mathbf{1}} \frac{\langle f, Lf \rangle}{\langle f, f \rangle} = \min_{f \perp \mathbf{1}} \frac{\frac{1}{d} \sum_{u \sim v} |f(u) - f(v)|^2}{\sum_{u \in V} f(u)^2}$$

Cheeger's inequality



Expansion: For a subset $S \subseteq V$, define

$$\phi(S) = \frac{|E(S)|}{d|S|}$$

$E(S)$ = set of edges with one endpoint in S.

k-way expansion constant:

$$\rho_G(k) = \min\{\max \phi(S_i) : S_1, S_2, \ldots, S_k \subseteq V \text{ disjoint}\}$$

Theorem [Cheeger70, Alon-Milman85, Sinclair-Jerrum89]:

$$\frac{\lambda_2}{2} \cdot \rho_G(2) \cdot \sqrt{2\lambda_2}$$

$\lambda_2 = 0 \iff G$ is disconnected.

Miclo's conjecture

$\lambda_k = 0 \iff G$ has at least k connected components.

$f_1, f_2, \ldots, f_k : V \to \mathbb{R}$ first k (orthonormal) eigenfunctions

spectral embedding: $F : V \to \mathbb{R}^k$

$$F(v) = (f_1(v), f_2(v), \ldots, f_k(v))$$

Higher-order Cheeger Conjecture [Miclo 08]:

For every graph G and $k \in \mathbb{N}$, we have

$$\frac{\lambda_k}{2} \cdot \quad \rho_G(k) \cdot \quad C(k)\sqrt{\lambda_k}$$

for some $C(k)$ depending only on k.

our results

$$\rho_G(k) = \min\{\max \phi(S_i) : S_1, S_2, \ldots, S_k \subseteq V \text{ disjoint}\}$$

Theorem: For every graph G and $k \in \mathbb{N}$, we have

$$\frac{\lambda_k}{2} \cdot \rho_G(k) \cdot O(k^2)\sqrt{\lambda_k}$$

Also, $\rho_G(k) \cdot O(\sqrt{\lambda_{2k} \log k})$

- actually, can put $\lambda_{(1+\delta)k}$ for any $\delta>0$
- tight up to this $(1+\delta)$ factor
- proved independently by [Louis-Raghavendra-Tetali-Vempala 11]

If G is planar (or more generally, excludes a fixed minor), then

$$\rho_G(k) \cdot O(\sqrt{\lambda_{2k}})$$

small set expansion

Corollary: For every graph G and $k \in \mathbb{N}$, there is a subset of vertices S such that $|S| \leq \frac{n}{2k}$ and,

$$\phi(S) \cdot \; O(\sqrt{\lambda_k \log k})$$

Previous bounds:

$$|S| \cdot \; \frac{n}{\sqrt{k}} \quad \textbf{and} \quad \phi(S) \cdot \; O(\sqrt{\lambda_k \log k})$$

[Louis-Raghavendra-Tetali-Vempala 11]

$$|S| \cdot \; \frac{n}{k^{0.01}} \quad \textbf{and} \quad \phi(S) \cdot \; O(\sqrt{\lambda_k \log_k n})$$

[Arora-Barak-Steurer 10]

Theorem: For every graph G and $k \in \mathbb{N}$, we have

$$\rho_G(k) \cdot \quad O(k^2)\sqrt{\lambda_k}$$

proof

Theorem: For every graph G and $k \in \mathbb{N}$, we have

$$\rho_G(k) \cdot \quad k^{O(1)} \sqrt{\lambda_k}$$

Dirichlet Cheeger inequality

For a mapping $F : V \to \ell_2$, define the **Rayleigh quotient:**

$$\mathcal{R}(F) = \frac{\frac{1}{\bar{d}} \sum_{u \sim v} \|F(u) - F(v)\|^2}{\sum_{u \in V} \|F(u)\|^2}$$

Lemma: For any mapping $F : V \to \ell_2$, there exists a subset

$$S \subseteq \mathrm{supp}(F) = \{v \in V : F(v) \neq 0\}$$

such that: $\phi(S) \leq \sqrt{2\mathcal{R}(F)}$

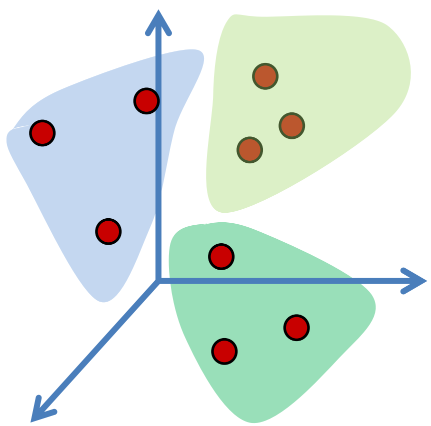
Miclo's disjoint support conjecture

Conjecture [Miclo 08]:

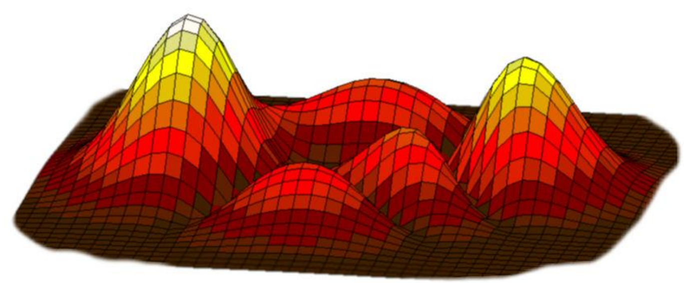
For every graph G and $\boldsymbol{k} \in \mathbb{N}$, there exist disjointly supported functions $\psi_1, \psi_2, \ldots, \psi_k : V \to \mathbb{R}$ so that for i=1, 2, ..., k,

$$\mathcal{R}(\psi_i) \le C(k)\lambda_k$$

Localizing eigenfunctions: $F(v) = (f_1(v), f_2(v), \ldots, f_k(v))$



$$\mathcal{R}(F) \le \lambda_k$$



$$\mathbb{R}^k$$

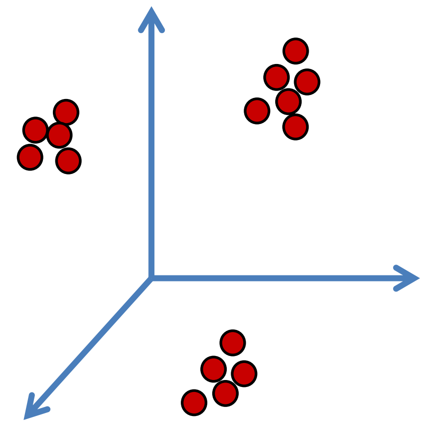
Isotropy: For every unit vector $x \in \mathbb{R}^k$

$$\sum_{v \in V} \langle x, F(v) \rangle^2 = 1$$

$$\|$$

$$x^T M M^T x = \|x\|^2$$

$$M = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix}$$



Total mass: $$\sum_{v \in V} \|F(v)\|^2 = k$$

radial projection

Define the **radial projection distance** on V by,

$$d_F(u,v) = \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\|$$

Fact: $\|F(u)\| \cdot d_F(u,v) \leq 2\,\|F(u) - F(v)\|$

Isotropy gives: For every subset $S \subseteq V$,

$$\mathrm{diam}(S, d_F) \cdot \quad \frac{1}{2} \implies \sum_{v \in S} \|F(v)\|^2 \cdot \quad \frac{2}{k} \sum_{v \in V} \|F(v)\|^2$$

Want to find k regions $S_1, S_2, \ldots, S_k \subseteq V$ such that,

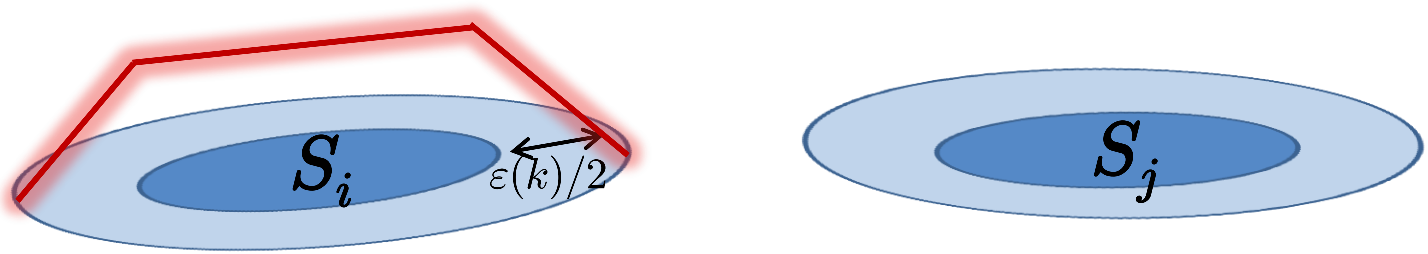
mass: $$\sum_{v \in S_i} \|F(v)\|^2 \asymp 1$$

separation: $$d_F(S_i, S_j) \geq \varepsilon(k) \text{ for all } i \neq j$$

Then define $\psi_i : V \to \mathbb{R}^k$ by,

$$\psi_i(v) = F(v) \cdot \max\left(0, 1 - \frac{2d_F(v, S_i)}{\varepsilon(k)}\right)$$

so that $\psi_1, \psi_2, \ldots, \psi_k$ are disjointly supported and $\psi_i|_{S_i} \equiv F|_{S_i}$



Want to find k regions S_1, S_2, ..., $S_k \subseteq V$ such that,

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Claim: $$\mathcal{R}(\psi_i) \leq \frac{O(k)}{\varepsilon(k)^2} \cdot \mathcal{R}(F)$$

Want to find k regions $S_1, S_2, \ldots, S_k \subseteq V$ such that,

mass: $$\sum_{v \in S_i} \|F(v)\|^2 \asymp 1$$

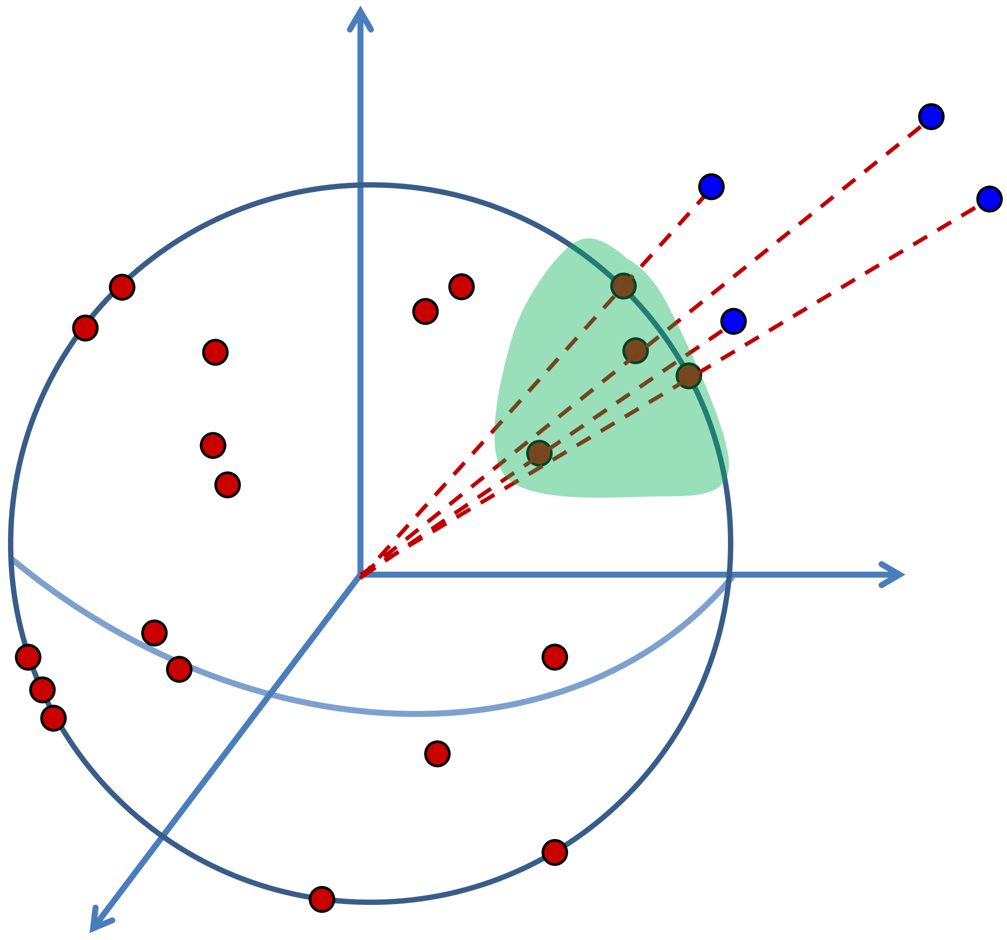
separation: $$d_F(S_i, S_j) \geq \varepsilon(k) \text{ for all } i \neq j$$

For every subset $S \subseteq V$,

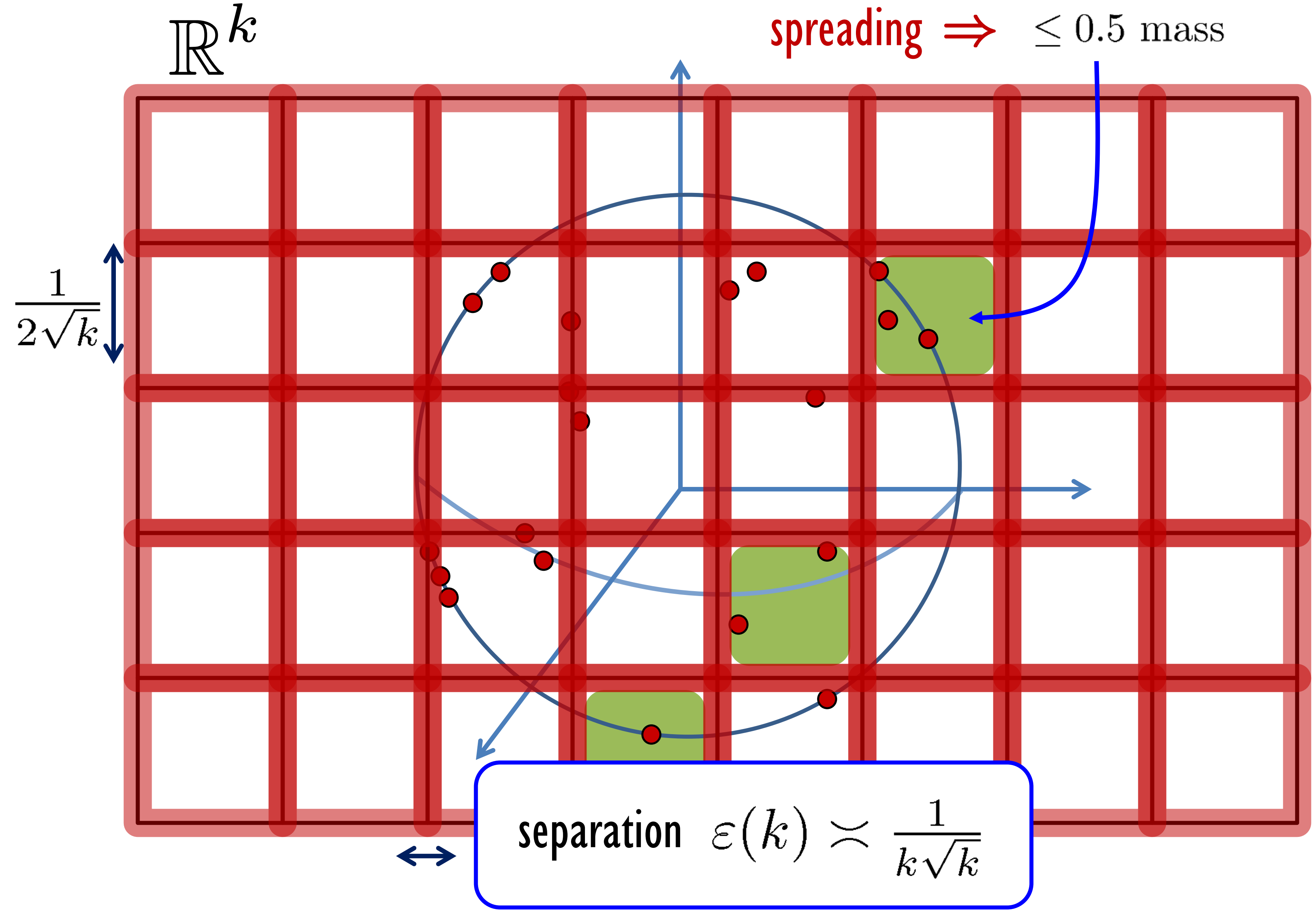
$$\operatorname{diam}(S, d_F) \cdot \quad \frac{1}{2} \implies \sum_{v \in S} \|F(v)\|^2 \cdot \quad \frac{2}{k} \sum_{v \in V} \|F(v)\|^2$$

How to break into subsets? Randomly...

$\mathbb{R}^k$



random partitioning



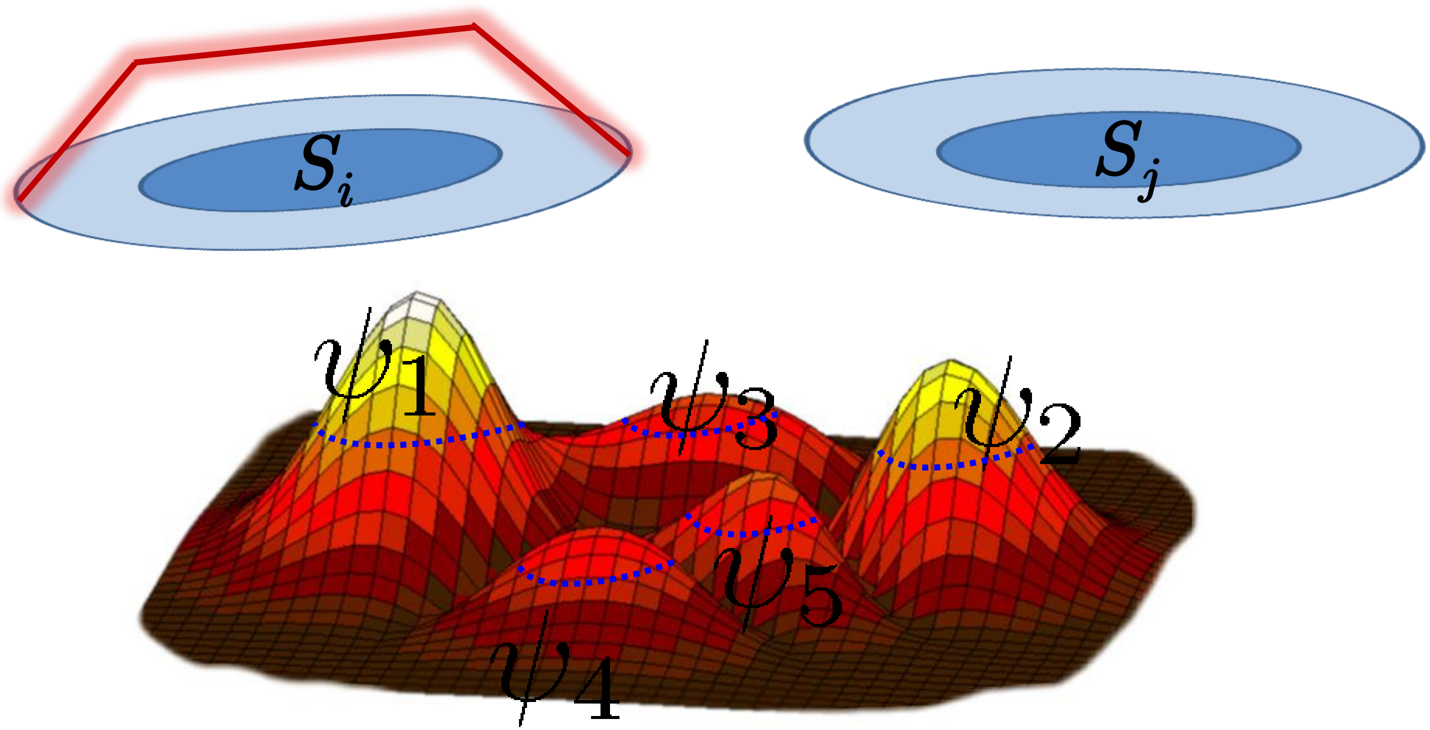
smooth localization

We found k regions $S_1, S_2, \ldots, S_k \subseteq V$ such that,

mass:

$$\sum_{v \in S_i} \|F(v)\|^2 \asymp 1$$

separation: $d_F(S_i, S_j) \geq \frac{1}{2k\sqrt{k}}$ for all $i \neq j$



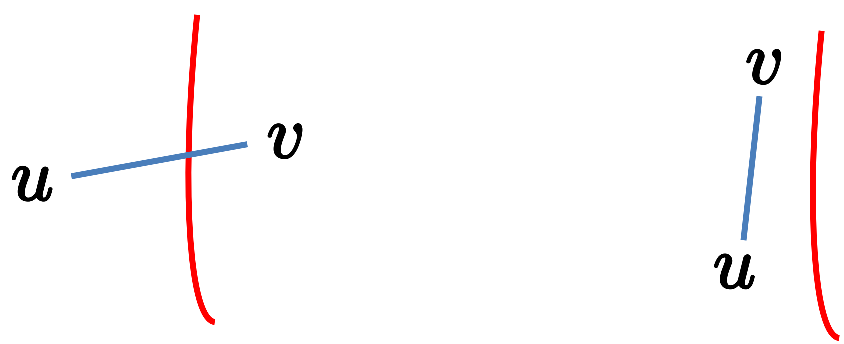
$$\rho_G(k) \cdot \quad O(\sqrt{\lambda_{2k} \log k})$$

- Take only best ***k/2*** regions (gains a factor of ***k***)
- Before partitioning, take a random projection into **O(log k)** dimensions

Recall the spreading property: For every subset $S \subseteq V$,

$$\mathrm{diam}(S, d_F) \cdot \quad \frac{1}{2} \implies \sum_{v \in S} \|F(v)\|^2 \cdot \quad \frac{2}{k} \sum_{v \in V} \|F(v)\|^2$$

- With respect to a random ball in d dimensions...



k-way spectral partitioning algorithm

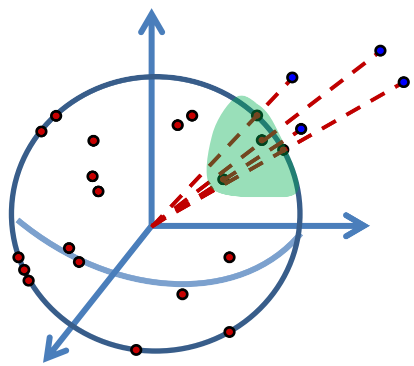
ALGORITHM:

1) Compute the spectral embedding

$$F(v) = (f_1(v), f_2(v), \ldots, f_k(v))$$

2) Partition the vertices according to the radial projection

$$d_F(u, v) = \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\|$$

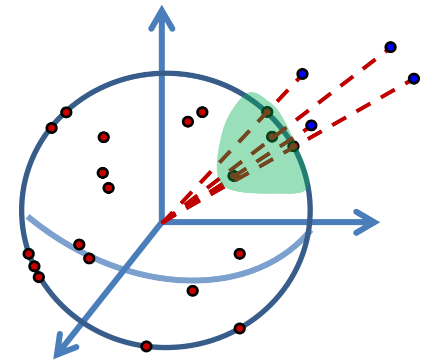


3) Peform a "Cheeger sweep" on each piece of the partition

planar graphs: spectral + intrinsic geometry

2) Partition the vertices according to the radial projection

$$d_F(u,v) = \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\|$$



For planar graphs, we consider the induced shortest-path metric on G, where an edge $\{u,v\}$ has length $d_F(u,v)$.

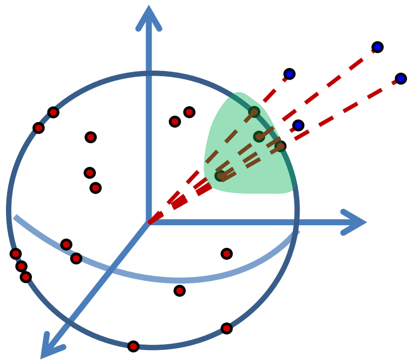
Now we can analyze the shortest-path geometry using

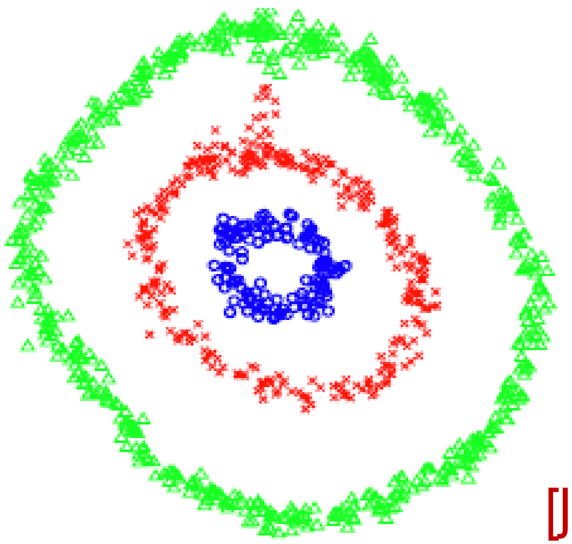
[Klein-Plotkin-Rao 93]

planar graphs: spectral + intrinsic geometry

2) Partition the vertices according to the radial projection

$$d_F(u,v) = \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\|$$





[Jordan-Ng-Weiss 01]

1) $\rho_G(k) \leq O(k^2)\sqrt{\lambda_k}$ Can this be made $\text{poly}(\log k)$?

2) Can [Arora-Barak-Steurer] be done geometrically?

We use k eigenvectors, find $\asymp k$ sets, lose $\sqrt{\log k}$.

What about using $\sqrt{n}$ eigenvectors to find $n^{0.01}$ sets?

3) **Small set expansion problem**

There is a subset $S \subseteq V$ with

$$|S| \leq \frac{n}{2k} \quad \text{and} \quad \phi(S) \leq O(\sqrt{\lambda_k \log k})$$

Tight for $k \leq \text{poly}(\log n)$ (noisy hypercubes)

Tight for $k \leq 2^{(\log n)^\varepsilon}$ (short code graph)

[Barak-Gopalan-Hastad-Meka-Raghvendra-Steurer 11]