Bratteli Diagrams and the Unitary Duals of Locally Finite Groups

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“Jersey Roots, Global Reach”

12th March 2012
Unitary Representations of Discrete Groups

Definition

If \( G \) is a countable group, then a unitary representation of \( G \) is a homomorphism \( \varphi : G \to U(H) \), where \( U(H) \) is the unitary group on the separable complex Hilbert space \( H \).

Definition

Two representations \( \varphi : G \to U(H) \) and \( \psi : G \to U(H) \) are unitarily equivalent if there exists \( A \in U(H) \) such that \( \psi(g) = A \varphi(g) A^{-1} \) for all \( g \in G \).

Definition

The unitary representation \( \varphi : G \to U(H) \) is irreducible if there are no nontrivial proper \( G \)-invariant closed subspaces \( 0 < W < H \).
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If $G$ is a countable group, then a unitary representation of $G$ is a homomorphism $\varphi : G \to U(\mathcal{H})$, where $U(\mathcal{H})$ is the unitary group on the separable complex Hilbert space $\mathcal{H}$. 

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Two representations $\varphi : G \to U(\mathcal{H})$ and $\psi : G \to U(\mathcal{H})$ are unitarily equivalent if there exists $A \in U(\mathcal{H})$ such that $\psi(g) = A \varphi(g) A^{-1}$ for all $g \in G$.

Definition

The unitary representation $\varphi : G \to U(\mathcal{H})$ is irreducible if there are no nontrivial proper $G$-invariant closed subspaces $0 < W < \mathcal{H}$. 
If $G$ is a countable group, then a unitary representation of $G$ is a homomorphism $\varphi : G \to U(\mathcal{H})$, where $U(\mathcal{H})$ is the unitary group on the separable complex Hilbert space $\mathcal{H}$.

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Definition

The unitary representation $\varphi : G \to U(\mathcal{H})$ is irreducible if there are no nontrivial proper $G$-invariant closed subspaces $0 < W < \mathcal{H}$. 
The irreducible unitary representations of $\mathbb{Z}$ are $\phi_z: \mathbb{Z} \rightarrow U^1(\mathbb{C}) = \{c \in \mathbb{C}: |c| = 1\}$ where $z \in T$ and $\phi_z(k)$ is multiplication by $z^k$.

The multiplicity-free unitary representations of $\mathbb{Z}$ can be parameterized by the Borel probability measures $\mu$ on $T$ so that the following are equivalent:

(i) the representations $\phi_{\mu}, \phi_{\nu}$ are unitarily equivalent;

(ii) the measures $\mu, \nu$ have the same null sets.
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(i) the representations $\varphi_\mu, \varphi_\nu$ are unitarily equivalent;
(ii) the measures $\mu, \nu$ have the same null sets.
Let $G$ be a countably infinite group. Let $H$ be a separable complex Hilbert space and let $U(H)$ be the corresponding unitary group. Then $U(H)$ is a Polish group and hence $U(H) G$ with the product topology is a Polish space.

The set $\text{Rep}(G) \subseteq U(H) G$ of unitary representations is a closed subspace and hence $\text{Rep}(G)$ is a Polish space.

The set $\text{Irr}(G)$ of irreducible representations is a $G$-δ subset of $\text{Rep}(G)$ and hence $\text{Irr}(G)$ is also a Polish space.
Let $G$ be a countably infinite group.
The Polish Space of Unitary Representations

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An equivalence relation $E$ on a Polish space $X$ is **Borel** if $E$ is a Borel subset of $X \times X$. 
Borel equivalence relations

Definition

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Theorem (Mackey)

The unitary equivalence relation $\approx_G$ on $\text{Irr}(G)$ is an $F_\sigma$ equivalence relation.

Theorem (Hjorth-Törnquist)

The unitary equivalence relation $\approx^+_G$ on $\text{Rep}(G)$ is an $F_{\sigma\delta}$ equivalence relation.
Definition (Mackey)

The Borel equivalence relation $E$ on the Polish space $X$ is **smooth** if there exists a Borel map $f : X \to \mathbb{R}$ such that

$$x \ E \ y \iff f(x) = f(y).$$

Theorem (Mackey)

Orbit equivalence relations arising from Borel actions of compact Polish groups on a Polish spaces are smooth.

Corollary

If $G$ is a countable group, then unitary equivalence for finite dimensional irreducible unitary representations of $G$ is smooth.
Smooth vs Nonsmooth

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Orbit equivalence relations arising from Borel actions of *compact* Polish groups on a Polish spaces are smooth.

**Corollary**

If $G$ is a countable group, then unitary equivalence for *finite dimensional* irreducible unitary representations of $G$ is smooth.
The Glimm-Thoma Theorem

Theorem (Glimm-Thoma)

If $G$ is a countable group, then the following are equivalent:

(i) $G$ is not abelian-by-finite.

(ii) $G$ has an infinite dimensional irreducible representation.

(iii) The unitary equivalence relation $\equiv_G$ on the space $\text{Irr}(G)$ of infinite dimensional irreducible unitary representations of $G$ is not smooth.

Question

Does this mean that we should abandon all hope of finding a "satisfactory classification" for the irreducible unitary representations of the other countable groups?
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Definition (Friedman-Kechris)

Let $E$, $F$ be Borel equivalence relations on the Polish spaces $X$, $Y$.

- $E \leq_B F$ if there exists a Borel map $\varphi : X \to Y$ such that
  $$x \ E \ y \iff \varphi(x) \ F \varphi(y).$$

In this case, $\varphi$ is called a **Borel reduction** from $E$ to $F$. 
Borel reductions

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- $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.
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- $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.

- $E <_B F$ if both $E \leq_B F$ and $E \not\sim_B F$. 

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12th March 2012
The Glimm-Effros Dichotomy

**Theorem (Harrington-Kechris-Louveau)**

If $E$ is a Borel equivalence relation on the Polish space $X$, then exactly one of the following holds:

(i) $E$ is smooth; or 

(ii) $E_0 \leq_B E$.

**Definition**

$E_0$ is the Borel equivalence relation on $2^\mathbb{N}$ defined by:

$$x \ E_0 \ y \iff x_n = y_n \text{ for all but finitely many } n.$$
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$E_0$ is the Borel equivalence relation on $2^\mathbb{N}$ defined by:

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Example

Baer’s classification of the rank 1 torsion-free abelian groups is essentially a Borel reduction to $E_0$. 
When it’s bad, it’s worse ...

Theorem (Hjorth 1997)

If the countable group G is not abelian-by-finite, then there exists a $U(H)$-invariant Borel subset $X \subseteq \text{Irr}(G)$ such that the unitary equivalence relation $\approx_{G \upharpoonright X}$ is turbulent.
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Remark

This is a much more serious obstruction to the existence of a “satisfactory classification” of the irreducible unitary representations of $G$. 
When it’s bad, it’s worse ...

Theorem (Hjorth 1997)

If the countable group $G$ is not abelian-by-finite, then there exists a $U(ℋ)$-invariant Borel subset $X \subseteq \text{Irr}(G)$ such that the unitary equivalence relation $\approx_G \upharpoonright X$ is turbulent.

Question (Dixmier-Effros-Thomas)

Do there exist countable groups $G, H$ such that

(i) $G, H$ are not abelian-by-finite; and

(ii) $\approx_G, \approx_H$ are not Borel bireducible?
When it’s bad, it’s worse ...

**Theorem (Hjorth 1997)**

*If the countable group $G$ is not abelian-by-finite, then there exists a $U(H)$-invariant Borel subset $X \subseteq \text{Irr}(G)$ such that the unitary equivalence relation $\approx_G \upharpoonright X$ is turbulent.*

**Conjecture (Thomas)**

*If $G$ is a nonabelian free group and $H$ is a “suitably chosen” amenable group, then $\approx_H \lt_B \approx_G$.***
Nonabelian free groups

Notation
\[ F_n \text{ denotes the free group on } n \text{ generators for } n \in \mathbb{N}^+ \cup \{\infty\}. \]

Observation
If \( G \) is any countable group, then \( \approx_G \) is Borel reducible to \( \approx_{F_\infty} \).
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If \( G \) is any countable group, then \( \approx_G \) is Borel reducible to \( \approx_{F_\infty} \).

Proof.
If \( \theta : F_\infty \rightarrow G \) is a surjective homomorphism, then the induced map

\[
\text{Irr}(G) \rightarrow \text{Irr}(F_\infty)
\]

\[
\varphi \mapsto \varphi \circ \theta
\]

is a Borel reduction from \( \approx_G \) to \( \approx_{F_\infty} \).
Theorem

$\approx_{F_{\infty}}$ is Borel reducible to $\approx_{F_2}$.

Sketch Proof.
If $f: \mathbb{N} \to \mathbb{N}$ be a suitably fast growing function, then we can induce representations from $F_{\infty} = \langle a, b \rangle_{|n \in \mathbb{N}}$ to the free group $F_2 = \langle a, b \rangle$.

Question
Does $H \leq G$ imply that $\approx_H$ is Borel reducible to $\approx_G$?
In particular, is $\approx_{F_2}$ Borel reducible to $\approx_{\text{SL}(3, \mathbb{Z})}$?
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Theorem

\[ \approx_{F_\infty} \text{ is Borel reducible to } \approx_{F_2}. \]

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If \( f : \mathbb{N} \to \mathbb{N} \) be a suitably fast growing function, then we can induce representations from

\[ F_\infty = \langle a^{f(n)} b a^{-f(n)} \mid n \in \mathbb{N} \rangle \leq N = \langle a^m b a^{-m} \mid m \in \mathbb{N} \rangle \]

to the free group \( F_2 = \langle a, b \rangle \).
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Question

\( H \leq G \) imply that \( \approx_H \) is Borel reducible to \( \approx_G \)?

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**Theorem**

\[ \cong_{F_{\infty}} \text{ is Borel reducible to } \cong_{F_{2}}. \]

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to the free group \( F_{2} = \langle a, b \rangle \).

**Question**

- Does \( H \leq G \) imply that \( \cong_{H} \) is Borel reducible to \( \cong_{G} \)?
- In particular, is \( \cong_{F_{2}} \) Borel reducible to \( \cong_{SL(3, \mathbb{Z})} \)?
A suitably chosen amenable group?

Definition

A countable group $G$ is amenable if there exists a left-invariant finitely additive probability measure $\mu : \mathcal{P}(G) \to [0, 1]$.

Some Candidates?

The direct sum $\bigoplus_{n \in \mathbb{N}} \text{Sym}(3)$ of countably many copies of $\text{Sym}(3)$.

A countably infinite extra-special $p$-group $P$; i.e. $P' = Z(P)$ is cyclic of order $p$ and $P/Z(P)$ is elementary abelian $p$-group.
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- A countably infinite **extra-special** $p$-group $P$; i.e. $P' = Z(P)$ is cyclic of order $p$ and $P/Z(P)$ is elementary abelian $p$-group.
The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

**Theorem**
Let $H$ be a countable locally finite group. If the countable group $G$ is not abelian-by-finite, then $\approx H$ is Borel reducible to $\approx G$.

**Corollary**
If $G, H$ are countable locally finite groups, neither of which is abelian-by-finite, then $\approx G$ and $\approx H$ are Borel bireducible.
The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

**Theorem**

Let $H$ be a countable locally finite group. If the countable group $G$ is not abelian-by-finite, then $\simeq_H$ is Borel reducible to $\simeq_G$. 

**Corollary**

If $G, H$ are countable locally finite groups, neither of which is abelian-by-finite, then $\simeq_G$ and $\simeq_H$ are Borel bireducible.
The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

**Theorem**

Let $H$ be a countable locally finite group. If the countable group $G$ is *not* abelian-by-finite, then $\approx_H$ is Borel reducible to $\approx_G$.

**Corollary**

If $G, H$ are countable locally finite groups, neither of which is abelian-by-finite, then $\approx_G$ and $\approx_H$ are Borel bireducible.
The reduced $C^*$-algebra

**Definition**

If $G$ is a countably infinite group, then the left regular representation

$$\lambda : G \to U(\ell^2(G))$$

extends to an injective $*$-homomorphism of the group algebra

$$\lambda : \mathbb{C}[G] \to \mathcal{L}(\ell^2(G)).$$

The **reduced $C^*$-algebra** $C^*_\lambda(G)$ is the completion of $\mathbb{C}[G]$ with respect to the norm $\|x\|_r = \|\lambda(x)\|_{\mathcal{L}(\ell^2(G))}$. 
The reduced $C^*$-algebra

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**Remark**

If $G$ is amenable, then there is a canonical correspondence between the irreducible representations of $G$ and $C^*_\lambda(G)$. 
Approximately finite dimensional $C^*$-algebras

**Definition**

A $C^*$-algebra $A$ is **approximately finite dimensional** if $A = \bigcup_{n \in \mathbb{N}} A_n$ is the closure of an increasing chain of finite dimensional sub-$C^*$-algebras $A_n$.

**Example**

If $G = \bigcup_{n \in \mathbb{N}} G_n$ is a locally finite group, then $C^*_\lambda(G) = \bigcup_{n \in \mathbb{N}} \mathbb{C}[G_n]$ is approximately finite dimensional.
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If $G = \bigcup_{n \in \mathbb{N}} G_n$ is a locally finite group, then $C^*_\lambda(G) = \overline{\bigcup_{n \in \mathbb{N}} \mathbb{C}[G_n]}$ is approximately finite dimensional.

**Remark**

Every finite dimensional $C^*$-algebra is isomorphic to a direct sum

$$\text{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \text{Mat}_{n_t}(\mathbb{C})$$

of full matrix algebras.
Theorem

If $G = \bigcup_{n \in \mathbb{N}} G_n$ is a locally finite group, then the following are equivalent:

(i) $G$ is not abelian-by-finite.

(ii) There exists a subsequence $(\ell_n | n \in \mathbb{N})$ and irreducible representations $\pi_n \in \text{Irr}(G_{\ell_n})$ such that for all $n \in \mathbb{N}$, $(\pi_n, \pi_{n+1}|_{G_{\ell_n}}) \geq 2$.

(iii) $\lim_{n \to \infty} \max\{\deg \pi | \pi \in \text{Irr}(G_n)\} = \infty$.

Question

Is there an "elementary" proof of this result?
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Elliot’s Theorem

Extending Glimm’s Theorem, Elliot proved:

**Theorem (Elliot 1977)**

If $\mathcal{A}$ is an approximately finite-dimensional $C^*$-algebra and $\mathcal{B}$ is a separable $C^*$-algebra such that $\cong_\mathcal{B}$ is non-smooth, then $\cong_\mathcal{A}$ is Borel reducible to $\cong_\mathcal{B}$.

**Corollary (Elliot 1977)**

If $\mathcal{A}, \mathcal{B}$ are approximately finite-dimensional $C^*$-algebras such that $\cong_\mathcal{A}, \cong_\mathcal{B}$ are non-smooth, then $\cong_\mathcal{A}$ and $\cong_\mathcal{B}$ are Borel bireducible.
Theorem (Sutherland 1983)

Let $H = \bigoplus_{n \in \mathbb{N}} \text{Sym}(3)$. If $G$ is any countable amenable group, then $\cong_G$ is Borel reducible to $\cong_H$. 

Corollary

If $G, H$ are countable amenable groups, neither of which is abelian-by-finite, then $\cong_G$ and $\cong_H$ are Borel bireducible.

Remark

The theorem ultimately depends upon the Ornstein-Weiss Theorem that if $G, H$ are countable amenable groups, then any free ergodic measure-preserving actions of $G, H$ are orbit equivalent.
Even less as expected ...

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Express $H = A \rtimes K$, where $A = \bigoplus_{n \in \mathbb{N}} C_3$ and $K = \bigoplus_{n \in \mathbb{N}} C_2$. 
Some representations of $H = \bigoplus_{n \in \mathbb{N}} \text{Sym}(3)$

- Express $H = A \rtimes K$, where $A = \bigoplus_{n \in \mathbb{N}} C_3$ and $K = \bigoplus_{n \in \mathbb{N}} C_2$.

- Then $\hat{A} = C_3^\mathbb{N}$ is the product of countably many copies of the cyclic group $C_3 = \{1, \xi, \xi^2\}$. 
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- Then $\hat{A} = C_3^\mathbb{N}$ is the product of countably many copies of the cyclic group $C_3 = \{1, \xi, \xi^2\}$.

- Let $Z = \{\xi, \xi^2\}^\mathbb{N} \subseteq \hat{A}$ and let $\mu$ be the usual product probability measure on $Z$. 
Some representations of $H = \bigoplus_{n \in \mathbb{N}} \text{Sym}(3)$

- Express $H = A \rtimes K$, where $A = \bigoplus_{n \in \mathbb{N}} C_3$ and $K = \bigoplus_{n \in \mathbb{N}} C_2$.

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- For each irreducible cocycle $\sigma : K \times Z \to U(\mathcal{H})$, there exists a corresponding irreducible representation

$$\pi_\sigma : H \to U(L^2(Z, \mathcal{H})).$$
Irreducible cocycles

If $\alpha, \beta : K \times \mathbb{Z} \to U(H)$ are cocycles, then $\text{Hom}(\alpha, \beta)$ consists of the Borel maps $b : \mathbb{Z} \to L(H)$ such that for all $g \in K$, $\alpha(g, x) b(x) = b(g \cdot x) \beta(g, x) \mu$-a.e.

The cocycle $\alpha$ is irreducible if $\text{Hom}(\alpha, \alpha)$ contains only scalar multiples of the identity.

The heart of the matter

If $K' \rtimes (\mathbb{Z}', \mu')$ is orbit equivalent to $K \rtimes (\mathbb{Z}, \mu)$, then the "cocycle machinery" is isomorphic via a Borel map.

Simon Thomas (Rutgers University)

Geometry and analysis of large networks

12th March 2012
If $\alpha, \beta : K \times \mathbb{Z} \to U(\mathcal{H})$ are cocycles, then $\text{Hom}(\alpha, \beta)$ consists of the Borel maps $b : \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ such that for all $g \in K$,

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The heart of the matter

If $K' \bowtie (Z', \mu')$ is orbit equivalent to $K \bowtie (Z, \mu)$, then the "cocycle machinery" is isomorphic via a Borel map.
Let $G$ be any countable amenable group and let $\Gamma = G \times \mathbb{Z}$. 

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Then the shift action of $\Gamma$ on $(X, \nu)$ is (essentially) free and strongly mixing.

For each irreducible representation $\varphi : G \to U(\mathcal{H})$, we can define an irreducible cocycle $\sigma_\varphi : (G \times \mathbb{Z}) \times X \to U(\mathcal{H})$ by

$$\sigma_\varphi(g, z, x) = \varphi(g)$$
Let $\text{Irr}(E_0)$ be the space of irreducible cocycles
\[ \sigma : K \times \mathbb{Z} \to U(\mathcal{H}) \]
and let $\approx_{E_0}$ be the equivalence relation defined by
\[ \sigma \approx_{E_0} \tau \iff \text{Hom}(\sigma, \tau) \neq 0. \]

If the countable group $G$ is amenable but not abelian-by-finite, then the unitary equivalence relation $\approx_G$ is Borel bireducible with $\approx_{E_0}$. 
Summing up ... 

Definition

Let $\text{Irr}(E_\infty)$ be the space of irreducible cocycles

$$\sigma : \mathbb{F}_2 \times 2^{\mathbb{F}_2} \to U(\mathcal{H})$$

and let $\approx_{E_\infty}$ be the equivalence relation defined by

$$\sigma \approx_{E_\infty} \tau \iff \text{Hom}(\sigma, \tau) \neq 0.$$ 

Theorem

The unitary equivalence relation $\approx_{\mathbb{F}_2}$ is Borel bireducible with $\approx_{E_\infty}$. 
Theorem

*If the countable group $G$ is amenable but not abelian-by-finite, then the unitary equivalence relation $\approx_G$ is Borel bireducible with $\approx_{E_0}$.*

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The unitary equivalence relation $\approx_{F_2}$ is Borel bireducible with $\approx_{E_\infty}$.

The Main Conjecture/Dream

$\approx_{E_\infty}$ is not Borel reducible to $\approx_{E_0}$.