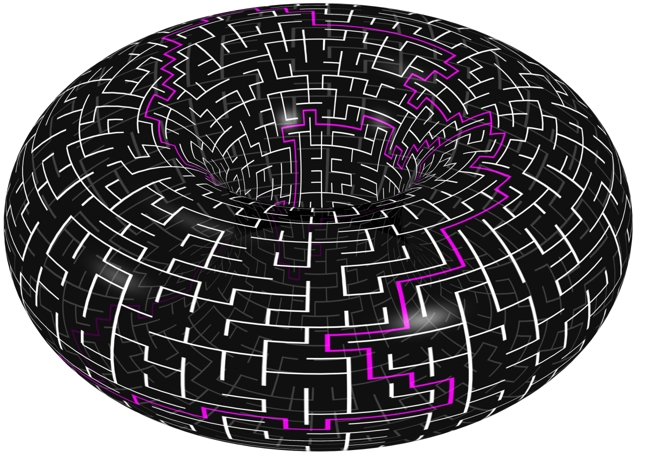
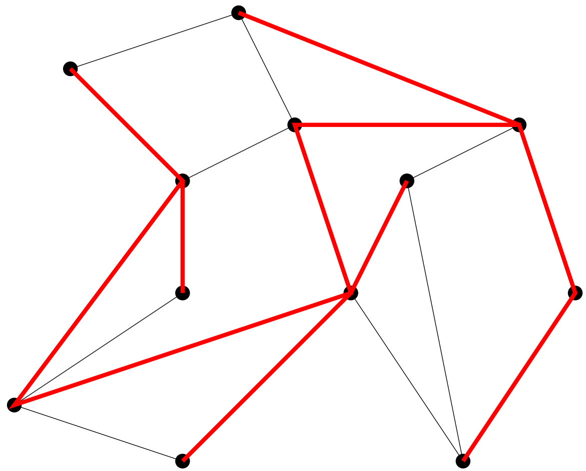
Random Complexes via Topologically-Inspired Determinants

BY RUSSELL LYONS

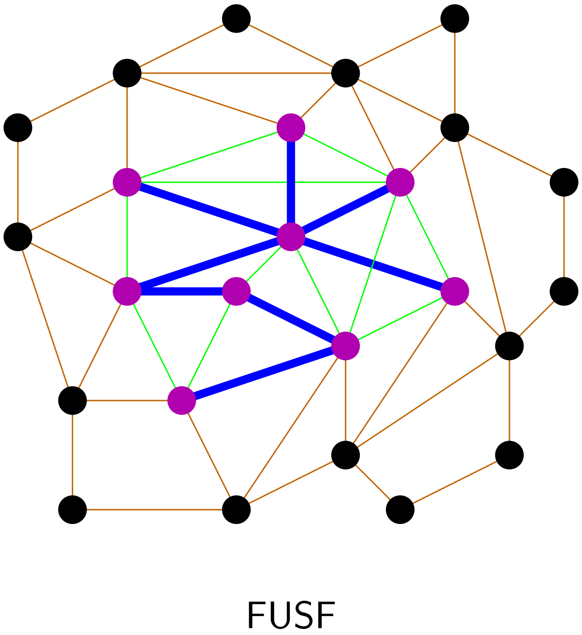
(Indiana University)

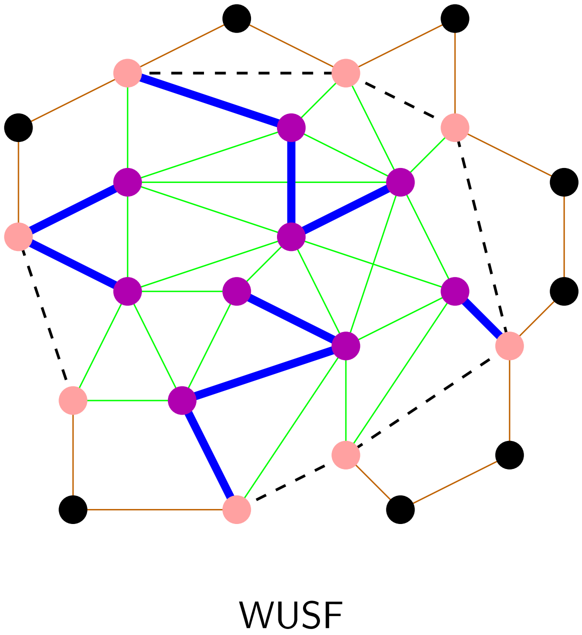


Uniform Spanning Trees

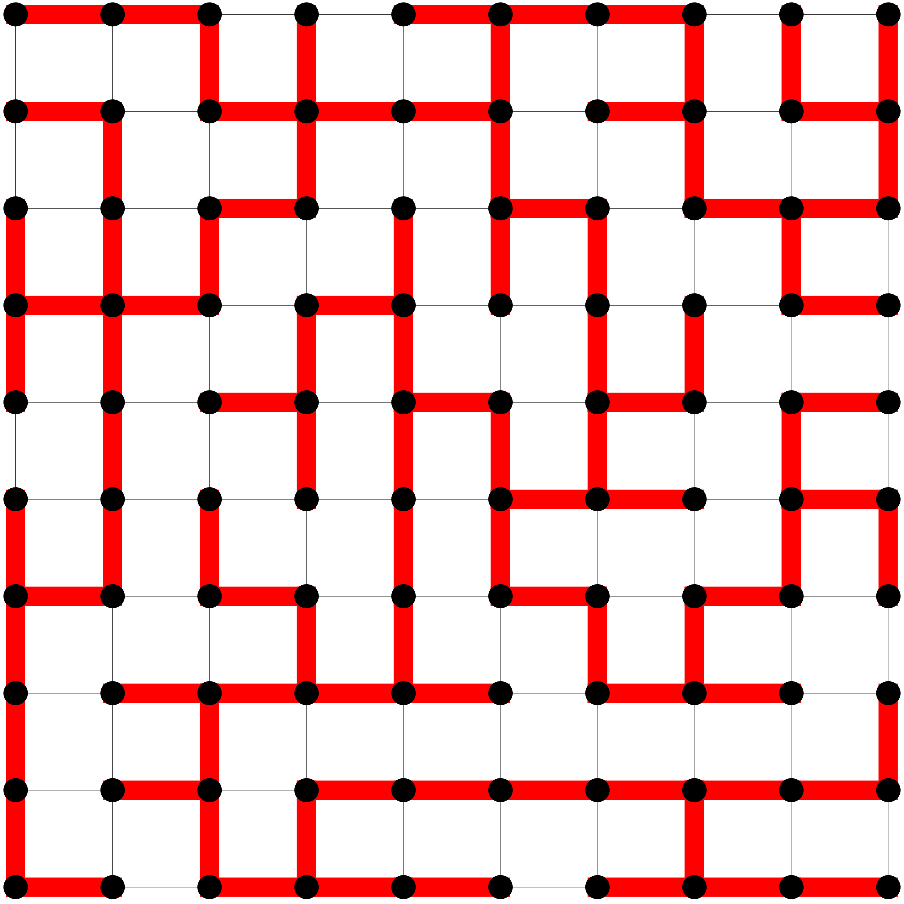


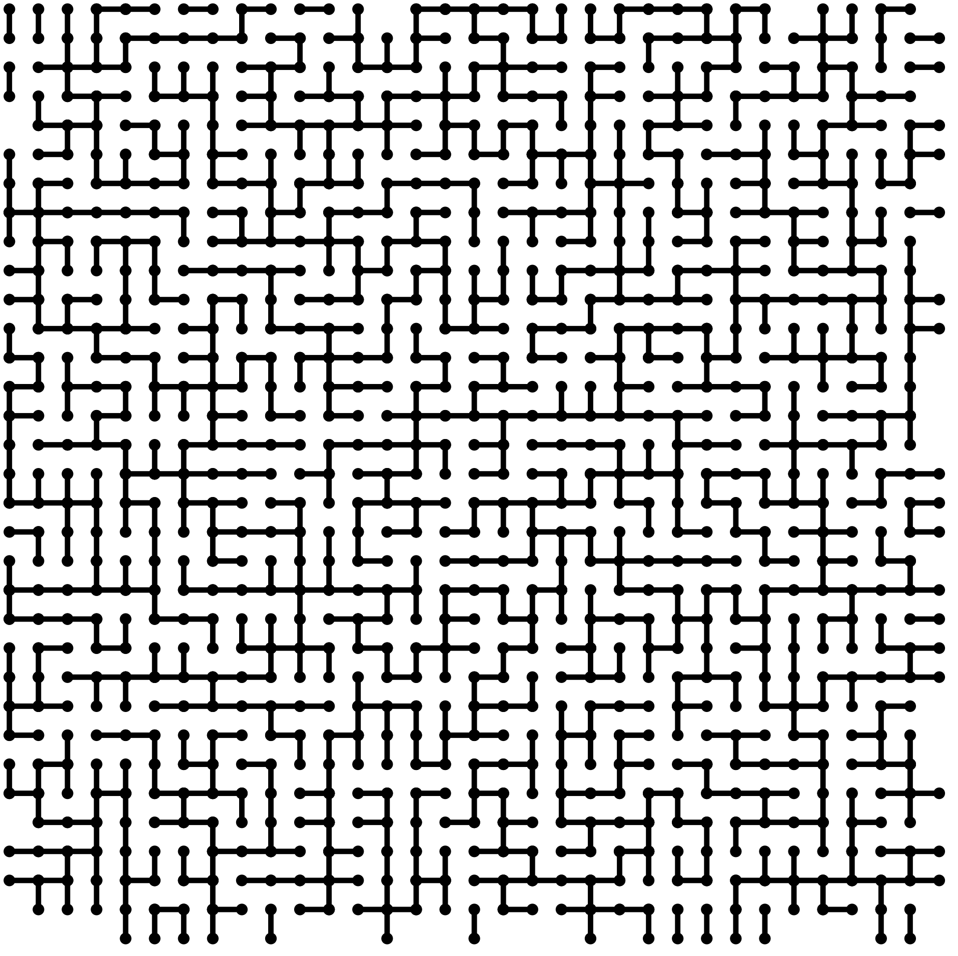
Algorithm of Aldous (1990) and Broder (1989): if you start a simple random walk at *any* vertex of a graph G and draw every edge it traverses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of G.





Pemantle (1991) showed that these weak limits of the uniform spanning tree measures always exist. These limits are now called the **free uniform spanning forest** on G and the **wired uniform spanning forest**. They are different, e.g., when G is itself a regular tree of degree at least 3.





(David Wilson)

Uniform Spanning Forests on $\mathbb{Z}^d$

Pemantle (1991) discovered the following interesting properties, among others:

- The free and the wired uniform spanning forest measures are the same on all euclidean lattices $\mathbb{Z}^d$.

- Amazingly, on $\mathbb{Z}^d$, the uniform spanning forest is a single tree a.s. if $d \leq 4$; but when $d \geq 5$, there are infinitely many trees a.s.

- If $2 \leq d \leq 4$, then the uniform spanning tree on $\mathbb{Z}^d$ has a single end a.s.; when $d \geq 5$, each of the infinitely many trees a.s. has at most two ends. Benjamini, Lyons, Peres, and Schramm (2001) showed that each tree has only one end a.s.

THEOREM (LYONS, PICHOT, AND VASSOUT, 2008). *Let G be a Cayley graph of a finitely generated infinite group Γ with respect to a finite generating set S. For every finite $K \subset \Gamma$, we have*

$$\frac{|\partial K|}{|K|} \ge 2\beta_1(\Gamma)\,.$$

In particular, this proves that finitely generated groups Γ with $\beta_1(\Gamma) > 0$ have uniform exponential growth. In fact, it shows uniform successive growth of balls, i.e., if

$$\bar{S} := \{\text{identity}\} \cup S \cup S^{-1}\,,$$

then

$$|\bar{S}^{n+1}|/|\bar{S}^{n}| \ge 1 + 2\beta_1(\Gamma)\,,$$

so

$$|\bar{S}^{n}| \ge [1 + 2\beta_1(\Gamma)]^{n}\,.$$

If E is finite and $H \subseteq \ell^2(E)$ is a subspace, it defines the determinantal measure

$$\forall T \subseteq E \text{ with } |T| = \dim H \qquad \mathbf{P}^H(T) := \det[P_H]_{T,T} \,,$$

where the subscript T, T indicates the submatrix whose rows and columns belong to T. This representation has a useful extension, namely,

$$\forall D \subseteq E \qquad \mathbf{P}^H[D \subseteq T] = \det[P_H]_{D,D} \,.$$

In case E is infinite and H is a closed subspace of $\ell^2(E)$, the determinantal probability measure $\mathbf{P}^H$ is defined via the requirement that this equation hold for all finite $D \subset E$.

Theorem (Lyons, 2003). *Let E be finite or infinite and let $H \subseteq H'$ be closed subspaces of $\ell^2(E)$. Then $\mathbf{P}^H \preccurlyeq \mathbf{P}^{H'}$, with equality iff $H = H'$.*

This means that there is a probability measure on the set $\{(T, T')\,;\ T \subseteq T'\}$ that projects in the first coordinate to $\mathbf{P}^H$ and in the second to $\mathbf{P}^{H'}$.

Let $G = (\mathsf{V}, \mathsf{E})$ be a finite graph. Choose one orientation for each edge $e \in \mathsf{E}$. Let $\bigstar = B^1(G)$ denote the subspace in $\ell^2(\mathsf{E})$ spanned by the stars (coboundaries) and let $\diamondsuit = Z_1(G)$ denote the subspace spanned by the cycles. Then $\ell^2(\mathsf{E}) = \bigstar \oplus \diamondsuit$.

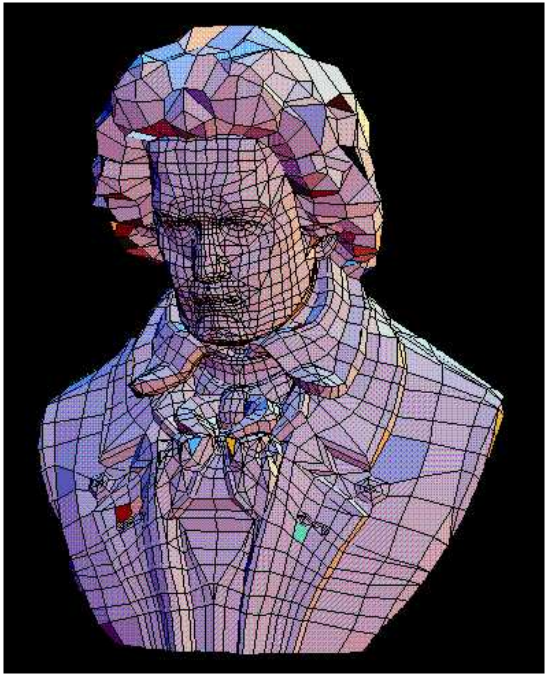
For a finite graph, Burton and Pemantle (1993) showed that the uniform spanning tree is the determinantal measure corresponding to orthogonal projection on $\bigstar = \diamondsuit^\perp$. (Precursors due to Kirchhoff (1847) and Brooks, Smith, Stone, and Tutte (1940).)

For an infinite graph, let $\bigstar := \bar{B}^1_c(G)$ be the closure in $\ell^2(\mathsf{E})$ of the span of the stars.

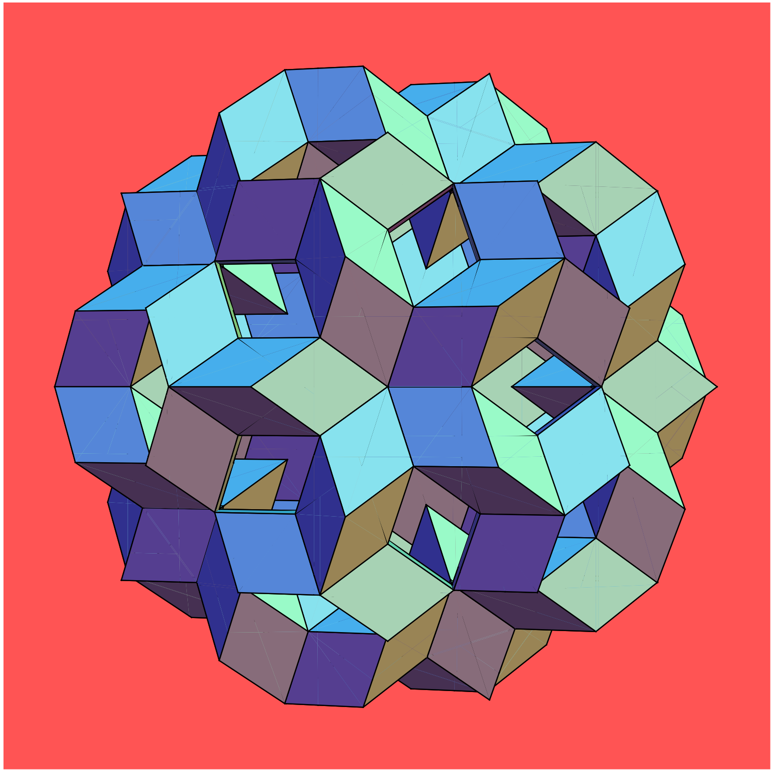
For an infinite graph, Benjamini, Lyons, Peres, and Schramm (2001) showed that WUSF is the determinantal measure corresponding to orthogonal projection on $\bigstar$, while FUSF is the determinantal measure corresponding to $\diamondsuit^\perp$.

Thus, $\mathsf{WUSF} \preccurlyeq \mathsf{FUSF}$, with equality iff $\bigstar = \diamondsuit^\perp$.

How do we extend the foregoing to higher dimensions? The higher-dimensional analogue of a graph is a CW-complex. A CW-complex is formed by sticking together cells:



(T. Robb via Mathematica)



Sándor Kabai and Lajos Szilassi via Mathematica

From Cayley to Kalai

What is the analogue of a spanning tree?

Cayley (1889) showed that the number of spanning trees in a complete graph on n vertices is n^{n-2}. Cayley's theorem was extended to higher dimensions by Kalai (1983), who showed that a certain enumeration of k-dimensional subcomplexes in a simplex on n vertices resulted in

$$n^{\binom{n-2}{k}}.$$

Kalai did not look at it this way, but we take the defining property of a spanning tree to be its property as a base of the graphical matroid, i.e., maximal without cycles.

Chain Groups and Bases for Finite CW-Complexes

Consider each cell of a CW-complex X to be oriented (except the 0-cells). Write $\Xi_k X$ for the set of k-cells of X. Identify cells with the corresponding basis elements of the chain and cochain groups, so that $\Xi_k X$ forms a basis of $C_k(X;\mathbb{R})$ and $C^k(X;\mathbb{R})$. The boundary map

$$\partial_k : C_k(X;\mathbb{R}) \to C_{k-1}(X;\mathbb{R})$$

has kernel $Z_k(X;\mathbb{R})$ and image $B_{k-1}(X;\mathbb{R})$, while the coboundary map

$$\delta_k = \partial_{k+1}^* : C^k(X;\mathbb{R}) \to C^{k+1}(X;\mathbb{R})$$

has kernel $Z^k(X;\mathbb{R})$ and image $B^{k+1}(X;\mathbb{R})$.

Given a finite CW-complex X and a subset $T \subseteq \Xi_k X$ of its k-cells, write X_T for the subcomplex

$$X_T := T \cup \bigcup_{j=0}^{k-1} \Xi_j X$$

We call T a k-**base** if it is maximal with $Z_k(X_T) = 0$.

Let X be a finite CW-complex. The determinantal probability measure $\mathbf{P}_k$ on the set of k-bases defined by orthogonal projection of $C_k(X)$ onto the space of coboundaries $B^k(X) = Z_k(X)^\perp$ is called the k**th lower matroidal measure on** X. If X is connected, then $\mathbf{P}_1$ is the law of the uniform spanning tree of the 1-skeleton of X. Let $t_j(T)$ denote the order of the torsion subgroup of $H_j(X_T; \mathbb{Z}) := Z_j(X_T; \mathbb{Z})/B_j(X_T; \mathbb{Z})$.

Proposition. *Let X be a finite CW-complex. For each k, there exists a_k such that for all k-bases T of X,*

$$\mathbf{P}_k(T) = a_k t_{k-1}(T)^2 \,.$$

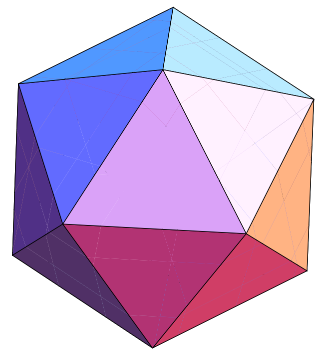
The theorem of Kalai (1983) is that when X is an $(n-1)$-dimensional simplex and $1 \le k \le n-1$,

$$\sum_T t_{k-1}(T)^2 = n^{\binom{n-2}{k}} \,,$$

where the sum is over all k-bases of X.

Example

Simplex on 6 vertices contains the projective plane, whose first homology group is $\mathbb{Z}_2$:



The projective plane can be embedded in $\mathbb{R}^4$.

LEMMA. *Let V be a subspace of $\mathbb{Q}^n$ of dimension r. Let $B_0 \subset V \cap \mathbb{Z}^n$ be a set of cardinality r that generates the group $V \cap \mathbb{Z}^n$. For any basis B of V that lies in $\mathbb{Z}^n$, identify B with the matrix whose columns are B in the standard basis of $\mathbb{Q}^n$ and write $\langle B \rangle$ for the subgroup of $\mathbb{Z}^n$ generated by B. Write $[G]$ for the torsion subgroup of a group G. Then for all such B, we have*

$$\det B^* B = |[\mathbb{Z}^n / \langle B \rangle]|^2 \det {B_0}^* B_0 \,.$$

Proof. By hypothesis on B_0, there exists an $r \times r$ integer matrix A such that $B = B_0 A$. We have

$$\det B^* B = \det A^* {B_0}^* B_0 A = \det A^* \det {B_0}^* B_0 \det A = (\det A)^2 \det {B_0}^* B_0 \,.$$

Also, $0 \to \langle B_0 \rangle / \langle B \rangle \to [\mathbb{Z}^n / \langle B \rangle] \to [\mathbb{Z}^n / \langle B_0 \rangle] \to 0$ is exact, whence

$$|[\mathbb{Z}^n / \langle B \rangle]| = |[\mathbb{Z}^n / \langle B_0 \rangle]| \cdot [\langle B_0 \rangle : \langle B \rangle] = [\langle B_0 \rangle : \langle B \rangle] = |\det A| \,.$$

Comparing these identities gives the result. ∎

Proof of Proposition

Given a set T of k-cells and S of $(k-1)$-cells, we write $\partial_{S,T}$ for the submatrix of ∂_k whose rows are indexed by S and columns by T. Now

$$\mathbf{P}_k(T) = \frac{\det \partial^*_{S,T}\partial_{S,T}}{\det \partial_{S,\Xi_k X}\partial^*_{S,\Xi_k X}}$$

for any fixed S indexing a basis of $B_k(X)$. If we multiply this formula by $\det \partial_{S,\Xi_k X}\partial^*_{S,\Xi_k X}$ and sum over S, then the Cauchy-Binet formula yields

$$\mathbf{P}_k(T) = \frac{\det \partial^*_{\Xi_{k-1}X,T}\partial_{\Xi_{k-1}X,T}}{\sum_S \det \partial_{S,\Xi_k X}\partial^*_{S,\Xi_k X}}\,.$$

That is, $\mathbf{P}_k(T)$ is proportional to

$$\det \partial^*_{\Xi_{k-1}X,T}\partial_{\Xi_{k-1}X,T}\,.$$

That is, $\mathbf{P}_k(T)$ is proportional to

$$\det \partial^*_{\Xi_{k-1}X,T}\partial_{\Xi_{k-1}X,T} .$$

Chain groups have integral coefficients for the duration of this proof. The columns of $\partial_{\Xi_{k-1}X,T}$ generate the group $B_{k-1}(X_T)$ and span the $\mathbb{Q}$-vector space $\mathbb{Q}^{\Xi_{k-1}X}$. Thus, the lemma shows that $\mathbf{P}_k(T)$ is proportional to $|[C_{k-1}(X_T)/B_{k-1}(X_T)]|^2$. Therefore, it suffices to show that

$$[C_{k-1}(X_T)/B_{k-1}(X_T)] = [Z_{k-1}(X_T)/B_{k-1}(X_T)]$$

in order to complete the proof. Let $u \in [C_{k-1}(X_T)/B_{k-1}(X_T)]$. Write

$$u = v + B_{k-1}(X_T)$$

with $v \in C_{k-1}(X_T)$. Let $n \in \mathbb{Z}^+$ be such that $nu = 0$, i.e., $nv \in B_{k-1}(X_T)$. Since $B_{k-1}(X_T) \subseteq Z_{k-1}(X_T)$, we have $\partial(nv) = 0$, which implies that $\partial v = 0$, i.e., that $v \in Z_{k-1}(X_T)$. Therefore $u \in [Z_{k-1}(X_T)/B_{k-1}(X_T)]$. ∎

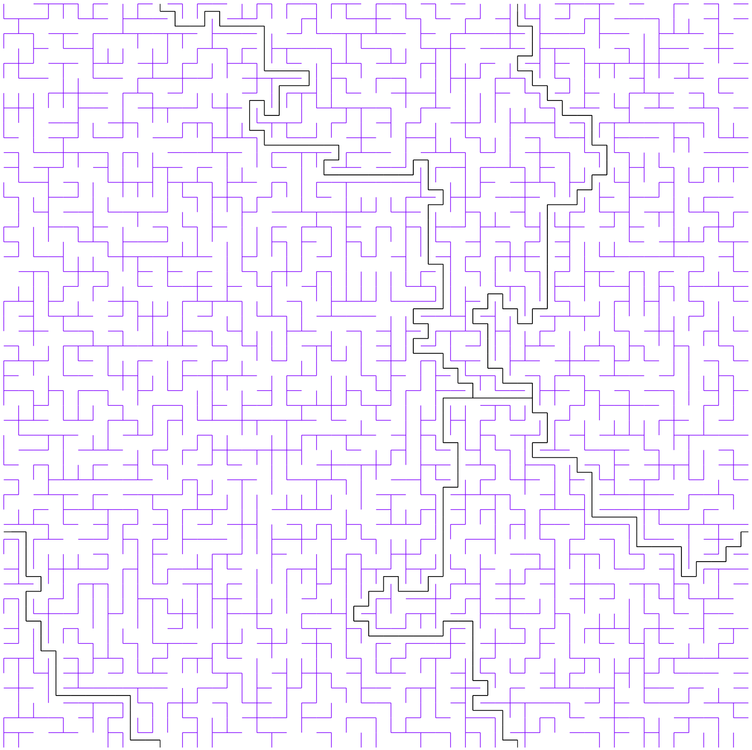
Upper Matroidal Measures

Another natural probability measure $\mathbf{P}^k$ on subsets of $\Xi_k X$ is the determinantal probability measure corresponding to the subspace of k-cocycles, $Z^k(X) = B_k(X)^\perp$. We call this measure the **kth upper matroidal measure on X**. Since $B^k(X) \subseteq Z^k(X)$, it follows that the upper measure $\mathbf{P}^k$ stochastically dominates the lower measure $\mathbf{P}_k$, with equality iff $H^k(X;\mathbb{R}) = 0$. As usual, let $b_k(X)$ denote the kth Betti number of X, the dimension of $H_k(X;\mathbb{R})$. One can add $b_k(X)$ k-cells to a sample from $\mathbf{P}_k$ to get a sample from $\mathbf{P}^k$.

Topological invariants for X reside in the *difference* between the measures.

Example

The torus:



Infinite Complexes

When X is infinite, there are natural extensions of the probability measures $\mathbf{P}_k$ and $\mathbf{P}^k$. We will usually assume that X is locally finite. In fact, the lower and upper measures each have two extensions, making four measures in all.

The k-cells form an orthonormal basis for the Hilbert space $C_k^{(2)}(X) := \ell^2(\Xi_k X)$, which is identified with its dual, the space of ℓ^2-cochains $C^k_{(2)}(X)$. Note: $C_k(X)$, $Z_k(X)$, and $B_k(X)$ denote spaces of k-chains with finite support. Let $C_c^k(X)$, $Z_c^k(X)$, $B_c^k(X)$ denote spaces of k-cochains with finite (compact) support.

measure $\leftrightsquigarrow$ subspace

$$\mathbf{P}^k_{\mathrm{W}} \leftrightsquigarrow \bar{Z}^k_c(X), \quad \mathbf{P}^k_{\mathrm{F}} \leftrightsquigarrow B_k(X)^{\perp}$$
$$\mathbf{P}_k^{\mathrm{W}} \leftrightsquigarrow \bar{B}^k_c(X), \quad \mathbf{P}_k^{\mathrm{F}} \leftrightsquigarrow Z_k(X)^{\perp}$$

$$\left\{ \begin{matrix} \text{free} \\ \text{wired} \end{matrix} \right\} \left\{ \begin{matrix} \text{upper} \\ \text{lower} \end{matrix} \right\} \text{ matroidal measures}$$

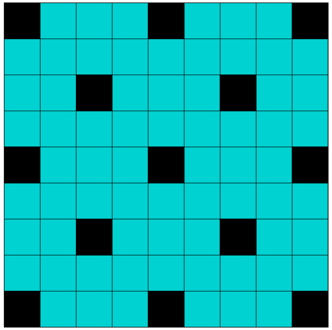
Examples

$$\mathbf{P}^{k}_{\mathrm{W}} \leftrightsquigarrow \bar{Z}^{k}_{c}(X), \quad \mathbf{P}^{k}_{\mathrm{F}} \leftrightsquigarrow B_{k}(X)^{\perp}$$
$$\mathbf{P}^{\mathrm{W}}_{k} \leftrightsquigarrow \bar{B}^{k}_{c}(X), \quad \mathbf{P}^{\mathrm{F}}_{k} \leftrightsquigarrow Z_{k}(X)^{\perp}$$

Always, $\mathbf{P}^{\mathrm{W}}_{1} = \mathsf{WUSF}$, while $\mathbf{P}^{\mathrm{F}}_{1} = \mathsf{FUSF}$.

In the cubical k-complex formed by the direct product $\mathbb{F}_2 \times \cdots \times \mathbb{F}_2$ of k free groups, $\mathbf{P}^{\mathrm{W}}_{k} \neq \mathbf{P}^{\mathrm{F}}_{k}$ (and the upper measures equal the lower).

Here, $\mathbf{P}_1 \neq \mathbf{P}^1$:



$$\mathbf{P}_{\mathrm{W}}^{k} \leftrightsquigarrow \bar{Z}_c^k(X), \quad \mathbf{P}_{\mathrm{F}}^{k} \leftrightsquigarrow B_k(X)^{\perp}$$
$$\mathbf{P}_k^{\mathrm{W}} \leftrightsquigarrow \bar{B}_c^k(X), \quad \mathbf{P}_k^{\mathrm{F}} \leftrightsquigarrow Z_k(X)^{\perp}$$

Since $B_c^k(X) \perp Z_k(X)$, we have $\bar{B}_c^k(X) \subseteq Z_k(X)^{\perp}$, whence $\mathbf{P}_k^{\mathrm{W}} \preccurlyeq \mathbf{P}_k^{\mathrm{F}}$. Since $Z_c^k(X) \subseteq B_k(X)^{\perp}$, we also have $\mathbf{P}_{\mathrm{W}}^{k} \preccurlyeq \mathbf{P}_{\mathrm{F}}^{k}$. Similarly, since $B_c^k(X) \subseteq Z_c^k(X)$, we have $\mathbf{P}_k^{\mathrm{W}} \preccurlyeq \mathbf{P}_{\mathrm{W}}^{k}$ and since $B_k(X) \subseteq Z_k(X)$, we have $\mathbf{P}_k^{\mathrm{F}} \preccurlyeq \mathbf{P}_{\mathrm{F}}^{k}$. Thus, all measures stochastically dominate the wired lower measure $\mathbf{P}_k^{\mathrm{W}}$, while all are dominated by the free upper measure $\mathbf{P}_{\mathrm{F}}^{k}$.

Hence, all four measures coincide iff $\mathbf{P}_k^{\mathrm{W}} = \mathbf{P}_{\mathrm{F}}^{k}$.

We have $H_k(X) = 0$ iff $Z_k(X) = B_k(X)$ iff $Z_k(X)^{\perp} = B_k(X)^{\perp}$ iff $\mathbf{P}_k^{\mathrm{F}} = \mathbf{P}_{\mathrm{F}}^{k}$. Likewise, $\bar{Z}_c^k(X) = \bar{B}_c^k(X)$ iff $\mathbf{P}_k^{\mathrm{W}} = \mathbf{P}_{\mathrm{W}}^{k}$, which is implied by (but is not equivalent to) $H_c^k(X) = 0$.

Suppose that Γ is a countable group acting freely on X by permuting the cells and the quotient X/Γ is compact. (Freeness here means that the stabilizer of each unoriented cell consists of only the identity of Γ.) In this case, we call X a **cocompact Γ-CW-complex**. Define $H_k^{(2)}(X) := Z_k^{(2)}(X)/\bar{B}_k^{(2)}(X)$, the reduced kth ℓ^2-homology group of X. The **kth ℓ^2-Betti number** of X is the von Neumann dimension of $H_k^{(2)}(X)$ with respect to Γ:

$$\beta_k(X;\Gamma) := \dim_\Gamma H_k^{(2)}(X)\,.$$

This is 0 iff $H_k^{(2)}(X) = 0$. The ℓ^2-Betti numbers of X are Γ-equivariant homotopy invariants of X by a theorem of Cheeger and Gromov (1986).

Amenable Groups

Recall that a countable group Γ is **amenable** if it has a **Følner exhaustion**, i.e., an increasing sequence of finite subsets V_n whose union is Γ such that for all finite $\mathsf{V} \subset \Gamma$, we have $\lim_{n\to\infty} |(\mathsf{V}\mathsf{V}_n)\triangle\mathsf{V}_n|/|\mathsf{V}_n| = 0$.

Suppose X is a Γ-CW-complex with finite fundamental domain D and Γ is amenable with Følner exhaustion $\langle \mathsf{V}_n \rangle$. Set $A_n := \mathsf{V}_n \bar{D}$. By a theorem of Dodziuk and Mathai (1998), we have

$$\lim_{n\to\infty} \frac{b_k(A_n)}{|\mathsf{V}_n|} = \beta_k(X;\Gamma)$$

for all k.

Using a new proof of this, we show that $\mathbf{P}_k^{\mathrm{W}} = \mathbf{P}_k^{\mathrm{F}}$ and $\mathbf{P}_{\mathrm{W}}^k = \mathbf{P}_{\mathrm{F}}^k$.
If $b_k(X) = 0$, then all 4 measures are equal.

Euclidean Complexes

Write $\mathbb{X}^d$ for the natural d-dimensional CW-complex determined by the hyperplanes of $\mathbb{R}^d$ passing through points of $\mathbb{Z}^d$ and parallel to the coordinate hyperplanes (so the 0-cells are the points of $\mathbb{Z}^d$).

Using the Euler-Poincaré formula, we show that the $\mathbf{P}_k$-probability that a given k-cell belongs to $\mathfrak{F}$ in $\mathbb{X}^d$ is k/d.

This is suggested by duality.

PROPOSITION. *Suppose that Γ is a countable group and X is a Γ-CW-complex with finite fundamental domain D. Then*

$$\mathbf{E}_{\mathrm{F}}^{k}\left[|\mathfrak{F} \cap D|\right] - \mathbf{E}_{k}^{\mathrm{W}}\left[|\mathfrak{F} \cap D|\right] = \beta_k(X; \Gamma)\,.$$

Again, ℓ^2-topological invariants for X reside in the *difference* between the measures.

COROLLARY. *If K is a $K(\Gamma, 1)$ CW-model with finite k-skeleton and X is its universal cover with fundamental domain D, then on X, we have*

$$\mathbf{E}_k^{\mathrm{F}}\big[|\mathfrak{F} \cap D|\big] - \mathbf{E}_k^{\mathrm{W}}\big[|\mathfrak{F} \cap D|\big] = \beta_k(\Gamma)\,.$$

Version for Cayley graphs:

PROPOSITION. *In every Cayley graph of a group Γ, we have*

$$\mathbf{E}_{\mathsf{FUSF}}[\deg_{\mathfrak{F}} o] = 2\beta_1(\Gamma) + 2\,.$$

For $A \subseteq X$, write $\mathsf{bnd}\, A$ for the **topological boundary** of A in X.

Corollary. *Fix $k \ge 1$. For a countable group Γ, every contractible Γ-CW-complex X with fundamental domain D and for which $\Xi_k X/\Gamma$ is finite satisfies*

$$\inf\left\{\frac{|\Xi_{k-1}\mathsf{bnd}\,(\mathsf{V}\bar{D})|}{|\mathsf{V}|}\,;\ \mathsf{V}\subset\Gamma \text{ is finite}\right\} \ge \beta_k(\Gamma)\,.$$

This gives an extension of a result of Lyons, Pichot, and Vassout (2008), which is the case $k = 1$ and $|\Xi_1 X/\Gamma| = 1$ (with slight differences).

In dimension 1, we can improve this inequality to be sharp as follows.

Theorem (Lyons, Pichot, and Vassout, 2008). *Let G be a Cayley graph of a finitely generated infinite group Γ with respect to a finite symmetric generating set S. For every finite $K \subset \Gamma$, we have*

$$\frac{|(KS)\setminus K|}{|K|} > 2\beta_1(\Gamma)\,.$$

Questions on $\mathbb{X}^d$

- What is the $(k-1)$-dimensional (co)homology of the random k-subcomplex? In the case $k = 1$ of spanning forests, this asks how many trees there are, the question answered by Pemantle (1991).

- If one takes the 1-point compactification of the random subcomplex, what is the k-dimensional (co)homology? In the case of spanning forests, this asks how many ends there are in the tree(s), the question answered partially by Pemantle (1991) and completely by Benjamini, Lyons, Peres, and Schramm (2001).

By translation-invariance of (co)homology and ergodicity of $\mathbf{P}_k$, we have that the values of the (co)homology groups are constants a.s. From the Alexander duality theorem, for $k = d-1$, we have $H_{k-1}(\mathfrak{F}) = 0$ $\mathbf{P}_k$-a.s., while $\mathbf{P}_k$-a.s., the Čech-Alexander-Spanier cohomology group $\check{H}^k(\mathfrak{F} \cup \infty)$ is 0 for $2 \le d \le 4$ and is a direct sum of infinitely many copies of $\mathbb{Z}$ for $d \ge 5$. It also follows from the Alexander duality theorem and from equality of free and wired limits that if $d = 2k$, then the a.s. values of $\check{H}^k(\mathfrak{F} \cup \infty)$ and $H_{k-1}(\mathfrak{F})$ are the same, so that the two bulleted questions above are dual in that case.

Alternative Formula

For subsets $A \subseteq [1, s]$, $B \subseteq E$, let $M_{A,B}$ denote the matrix determined by the rows of M indexed by A and the columns of M indexed by B. Let H be the row space of M. One definition of the determinantal probability measure $\mathbf{P}^H$ corresponding to M is

$$\mathbf{P}^H(T) = |\det M_{A,T}|^2 / \det \big(M_{A,E}(M_{A,E})^*\big)$$

whenever the rows indexed by A span H, where the superscript $*$ denotes adjoint. (One way to see that this defines a probability measure is to use the Cauchy-Binet formula.)

Suppose that X is the 2-complex defined by a connected graph G embedded in the 2-torus, all of whose faces are contractible. Let $G^\dagger$ be the graph dual to G. Then $\mathbf{P}_0$ is concentrated on the empty set, while $\mathbf{P}^0$ is the law of a uniform random vertex of G. The uniform spanning tree of G has law $\mathbf{P}_1$, while the edges of G that do not cross a uniform spanning tree of $G^\dagger$ have law $\mathbf{P}^1$. If $T \sim \mathbf{P}^1$, then T has non-contractible cycles, but no contractible cycles. The edges of such a T generate the homology $\mathbb{Z}^2$ of the 2-torus. This duality is shown in the random sample of the figure, where the lavender edges have law $\mathbf{P}_1$ on a 50×50 square lattice graph G, and those edges belonging to a cycle in $G^\dagger$ for $\mathbf{P}^1$ are shown in black, the other edges not being shown at all. Finally, $\mathbf{P}_2$ is the law of the complement of a uniform random face of G and $\mathbf{P}^2$ is concentrated on the full set of all 2-cells of X. We conjecture that the expected number of edges that belong to a cycle for the law $\mathbf{P}^1$ on an $n \times n$ square graph is asymptotic to $Cn^{5/4}$ for some constant C; cf. Kenyon (2000).

Cells per Vertex

Denote the number of k-cells in X/Γ by $f_k = f_k(X/\Gamma)$. Suppose that Γ is amenable and $b_k(X) = 0$. Then all four measures coincide. Write $\mathfrak{F}$ for a sample from $\mathbf{P}_k$. Using the Euler-Poincaré formula, one can show that if the k-skeleton is cocompact, then the $\mathbf{P}_k$-expected number of k-cells in $\mathfrak{F}$ per vertex of X equals

$$f_{k-1}/f_0 + \sum_{j=0}^{k-2} (-1)^{k+j-1} \big(f_j - \beta_j(X;\Gamma)\big)/f_0 \,.$$

This also equals the average number of k-cells in $\mathfrak{F}$ per vertex of X $\mathbf{P}_k$-a.s.

In this case of $\mathbb{X}^d$, we have $f_j = \binom{d}{j}$ and $\beta_j(\mathbb{X}^d;\mathbb{Z}^d) = 0$, whence the $\mathbf{P}_k$-expected number of k-cells per vertex equals $\binom{d-1}{k-1}$. Since the number of k-cells of $\mathbb{X}^d$ per vertex is $\binom{d}{k}$ and all k-cells have the same probability by symmetry, the $\mathbf{P}_k$-probability that a given k-cell belongs to $\mathfrak{F}$ in $\mathbb{X}^d$ is k/d.

PROPOSITION. *Suppose that* Γ *is a countable group and* X *is a* Γ*-CW-complex with finite fundamental domain* D*. Let* H *be a* Γ*-invariant closed subspace of* $C_k^{(2)}(X)$*. Then*

$$\mathbf{E}^{H}\left[|\mathfrak{F} \cap D|\right] = \dim_{\Gamma} H\,.$$

In particular,

$$\mathbf{E}_{\mathrm{F}}^{k}\left[|\mathfrak{F} \cap D|\right] - \mathbf{E}_{k}^{\mathrm{W}}\left[|\mathfrak{F} \cap D|\right] = \beta_k(X;\Gamma)\,.$$

Proof. Let the standard basis elements of $C_k^{(2)}(X)$ be $\{f_{\gamma,e}\,;\ \gamma \in \Gamma, e \in \Xi_k D\}$. Let o be the identity of Γ. Then

$$\mathbf{E}^{H}\left[|\mathfrak{F} \cap D|\right] = \sum_{e \in \Xi_k D} \mathbf{P}^{H}\left[e \in \mathfrak{F}\right] = \sum_{e \in \Xi_k D} (P_H f_{o,e},\, f_{o,e}) = \dim_{\Gamma} H\,. \qquad \blacksquare$$

A complex K is called a $K(\Gamma, 1)$ **CW-model** if K is a CW-complex with fundamental group equal to Γ and vanishing higher homotopy groups. If X is the universal cover of K and the k-skeleton of K is finite, we define $\beta_k(\Gamma) := \beta_k(X; \Gamma)$; it depends only on Γ and not on K.

Corollary. *If K is a $K(\Gamma, 1)$ CW-model with finite k-skeleton and X is its universal cover with fundamental domain D, then on X, we have*

$$\mathbf{E}_k^{\mathrm{F}}\left[|\mathfrak{F} \cap D|\right] - \mathbf{E}_k^{\mathrm{W}}\left[|\mathfrak{F} \cap D|\right] = \beta_k(\Gamma)\,.$$

Proof. Since the higher homotopy groups of X also vanish, so do its homology groups. Thus, $\mathbf{P}_k^{\mathrm{F}} = \mathbf{P}_{\mathrm{F}}^k$. By definition, $\beta_k(\Gamma) = \beta_k(X; \Gamma)$. ∎

Version for Cayley graphs:

Proposition. *In any Cayley graph of a group Γ, we have*

$$\mathbf{E}_{\mathsf{FUSF}}[\deg_{\mathfrak{F}} o] = 2\beta_1(\Gamma) + 2\,.$$

For $A \subseteq X$, write $\mathsf{bnd}\, A$ for the **topological boundary** of A in X.

PROPOSITION. *Suppose that* Γ *is a countable group and* X *is a* Γ*-CW-complex whose* k*-skeleton is cocompact for some fixed* $k \geq 1$*. Let* D *be a fundamental domain for the action of* Γ *on* X*. If* $b_k(X) = 0$*, then*

$$\inf \left\{ \frac{|\Xi_{k-1} \mathsf{bnd}\, (\mathsf{V}\bar{D})|}{|\mathsf{V}|} \; ; \; \mathsf{V} \subset \Gamma \textit{ is finite} \right\} \geq \beta_k(X; \Gamma)\,.$$

von Neumann dimension

If $H \subseteq \ell^2(\Gamma)$ is invariant under Γ, then $\dim_\Gamma H$ is a notion of dimension of H per element of Γ: If Γ is finite, then it is just $(\dim H)/|\Gamma| = (\mathrm{tr} P_H)/|\Gamma|$. In general, it is the common diagonal element of the matrix of P_H. More generally, if $H \subseteq \ell^2(\Gamma)^n$ is Γ-invariant, then $\dim_\Gamma H$ is the trace of the common diagonal $n \times n$ block element of the matrix of P_H.

Example: Let $\Gamma := \mathbb{Z}$, so that $\ell^2(\mathbb{Z}) \cong L^2[0,1]$ and H becomes $L^2(A)$ for $A \subseteq [0,1]$. Then $\dim_{\mathbb{Z}} H = |A|$ since $P_{L^2(A)} f = f \mathbf{1}_A$, so $\dim_{\mathbb{Z}} H = \int_0^1 (\mathbf{1} \mathbf{1}_A) \mathbf{1} = |A|$.

When $H \subseteq \ell^2(\Gamma)^n$ is Γ-invariant, the probability measure $\mathbf{P}^H$ on subsets of Γ^n is Γ-invariant.

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