Expanders and Baum-Connes type conjectures

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Coarse disjoint unions

Want to study the coarse Baum-Connes conjecture for spaces X as follows.
Let (X_n, d_n) be a sequence of finite graphs with edge metric.
Assume $3 \leq deg(v) \leq M$ for all vertices v.
Their *coarse disjoint union* is $X = \sqcup X_n$ equipped with any metric d such that:

- d restricts to d_n on X_n;
- $d(X_n, X_m) \to \infty$ as $n, m \to \infty$ $(n \neq m)$.

Examples:

- Box spaces: $X_n = \Gamma/\Gamma_n$, Γ a finitely generated group.
- Expanders (generic among such sequences).

Expanders

X_n: finite connected graph. $\Delta_n : l^2(X_n) \to l^2(X_n)$ graph Laplacian:

$$\Delta_n \ : \ \delta_x \mapsto \sum_{d(x,y)=1} \delta_x - \delta_y .$$

A sequence (X_n) is an expander if there exists $\epsilon > 0$ such that

$$\text{spectrum}(\Delta_n) \subseteq \{0\} \sqcup [\epsilon, 2M]$$

for all n.
Example: box spaces of property (T) groups.

Expanders and Coarse Baum-Connes

Set

$$\Delta = \oplus_n \Delta_n \ : \ l^2(X) \to l^2(X), \qquad \Delta \in \mathbb{C}_u[X]$$

and

$$p = \lim_{t\to\infty} e^{-t\Delta} \in C_u^*(X).$$

p is a *ghost projection*:

$$p_{x,y} \to 0 \quad \text{as } (x,y) \to \infty.$$

There are two traces $tr, \tau : C_u^*(X) \to \mathbb{R}$ such that
$\tau([p]) = 0, \ tr([p]) = 1$...
but index theory $\Rightarrow$ tr, τ agree on the range of assembly.

Three expansion conditions

$\Delta_n : l^2_0(X_n) \to l^2_0(X_n)$ ('$_0$' : no constants), $\Delta = \oplus_n \Delta_n$.

- Expanders: $\exists \epsilon$ such that $\langle \Delta_n \xi, \xi \rangle \geq \epsilon$.
- Weak expanders: $\exists \epsilon$ such that for all S and all $n \geq N_S$, if diam(supp(ξ)) $\leq S$, then $\langle \Delta_n \xi, \xi \rangle \geq \epsilon$.
- Strong expanders: $\exists \epsilon$ such that if $\langle \Delta \xi, \xi \rangle \geq \epsilon$ in *any* representation of $\mathbb{C}_u[X]$ with 'no constants'.

Examples and comments

- (X_n) with $\text{girth}(X_n) \to \infty$. Always weak expanders, sometimes expanders, never strong expanders.
- A box space associated to Γ is a strong expander if and only if Γ has property (T).
- Weak expander implies not property A; expander implies not coarsely embeddable.

Open question: 'geometric' characterisation, or even examples, of strong expanders?

Coarse Baum-Connes and girth

Let (X_n) be such that girth$(X_n) \to \infty$. (Uniform) coarse Baum-Connes assembly map:

$$\mu \ : \ \lim_r K_*^u(P_r(X)) \to K_*(C_u^*(X)).$$

Theorem

1. *μ is injective;*
2. *if X is an expander, μ is not surjective;*
3. *μ_{max} is an isomorphism.*

Corollary: Odd behaviour for Gromov monster groups.

Theorem

Part (3) always fails for strong expanders.

Questions and comments

- Finn-Sell, Wright: simpler / more conceptual proofs.
- Expanders with large girth are fairly well-behaved. Do they coarsely embed into any Banach space with 'half-way reasonable' properties (e.g. uniformly convex, *property (H)* of Kasparov-Yu)?
- Conjecture: for any $p > 2$ there exists a sequence of graphs with large girth that coarsely embeds into l^p but not l^q for any $q < p$.
- Can one embed an expander with geometric (T) into a f.p. group?
- Can one give a good 'geometric' criterion for recognising an expander with geometric (T) (preferably one that extends to connected graphs)?