Expanders and Baum-Connes type conjectures

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Expanders and coarse Baum-Connes

Some ‘types’ of expanders

More on expanders and coarse Baum-Connes
Coarse disjoint unions

Want to study the coarse Baum-Connes conjecture for spaces $X$ as follows.
Let $(X_n, d_n)$ be a sequence of finite graphs with edge metric.
Assume $3 \leq \deg(v) \leq M$ for all vertices $v$.
Their coarse disjoint union is $X = \bigsqcup X_n$ equipped with any metric $d$
such that:

- $d$ restricts to $d_n$ on $X_n$;
- $d(X_n, X_m) \to \infty$ as $n, m \to \infty$ ($n \neq m$).

Examples:

- Box spaces: $X_n = \Gamma/\Gamma_n$, $\Gamma$ a finitely generated group.
- Expanders (generic among such sequences).
Expanders

\(X_n\): finite connected graph. \(\Delta_n : l^2(X_n) \rightarrow l^2(X_n)\) graph Laplacian:

\[
\Delta_n : \delta_x \mapsto \sum_{d(x,y)=1} \delta_x - \delta_y.
\]

A sequence \((X_n)\) is an expander if there exists \(\epsilon > 0\) such that

\[
spectrum(\Delta_n) \subseteq \{0\} \cup [\epsilon, 2M]
\]

for all \(n\).

Example: box spaces of property (T) groups.
Expanders and Coarse Baum-Connes

Set

\[ \Delta = \bigoplus_n \Delta_n : l^2(X) \to l^2(X), \quad \Delta \in \mathbb{C}_u[X] \]

and

\[ p = \lim_{t \to \infty} e^{-t\Delta} \in C^*_u(X). \]

\( p \) is a ghost projection:

\[ p_{x,y} \to 0 \quad \text{as} \quad (x, y) \to \infty. \]

There are two traces \( tr, \tau : C^*_u(X) \to \mathbb{R} \) such that

\[ \tau([p]) = 0, \quad tr([p]) = 1 \ldots \]

but index theory \( \Rightarrow \) \( tr, \tau \) agree on the range of assembly.
Three expansion conditions

\[ \Delta_n : l_0^2(X_n) \rightarrow l_0^2(X_n) \quad (\text{‘0’ : no constants}), \quad \Delta = \bigoplus_n \Delta_n. \]

- **Expanders**: \( \exists \epsilon \) such that \( \langle \Delta_n \xi, \xi \rangle \geq \epsilon. \)

- **Weak expanders**: \( \exists \epsilon \) such that for all \( S \) and all \( n \geq N_S \), if \( \text{diam}(\text{supp}(\xi)) \leq S \), then \( \langle \Delta_n \xi, \xi \rangle \geq \epsilon. \)

- **Strong expanders**: \( \exists \epsilon \) such that if \( \langle \Delta \xi, \xi \rangle \geq \epsilon \) in *any* representation of \( C_u[X] \) with ‘no constants’. 
Examples and comments

- \((X_n)\) with \(\text{girth}(X_n) \rightarrow \infty\). Always weak expanders, sometimes expanders, never strong expanders.

- A box space associated to \(\Gamma\) is a strong expander if and only if \(\Gamma\) has property \((T)\).

- Weak expander implies not property A; expander implies not coarsely embeddable.

Open question: ‘geometric’ characterisation, or even examples, of strong expanders?
Coarse Baum-Connes and girth

Let \((X_n)\) be such that \(\text{girth}(X_n) \to \infty\). (Uniform) coarse Baum-Connes assembly map:

\[
\mu : \lim_r K^u_r(P_r(X)) \to K_*(C^*_u(X)).
\]

Theorem

1. \(\mu\) is injective;
2. if \(X\) is an expander, \(\mu\) is not surjective;
3. \(\mu_{\text{max}}\) is an isomorphism.

Corollary: Odd behaviour for Gromov monster groups.

Theorem

Part (3) always fails for strong expanders.
Questions and comments

- Finn-Sell, Wright: simpler / more conceptual proofs.
- Expanders with large girth are fairly well-behaved. Do they coarsely embed into any Banach space with ‘half-way reasonable’ properties (e.g. uniformly convex, property (H) of Kasparov-Yu)?
- Conjecture: for any $p > 2$ there exists a sequence of graphs with large girth that coarsely embeds into $l^p$ but not $l^q$ for any $q < p$.
- Can one embed an expander with geometric (T) into a f.p. group?
- Can one give a good ‘geometric’ criterion for recognising an expander with geometric (T) (preferably one that extends to connected graphs)?