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QUADRATIC ISOMETRIC FUNCTIONS OF THE HEISENBERG GROUPS. A COMBINATORIAL PROOF

A. Yu. Olshanskii and M. V. Sapir 512.543

Quadratic isoperimetric inequalities for Heisenberg groups $\mathcal{H}^{2n+1}$ have been announced in [2] and [4] and proved in [1]. The proof in [1] uses techniques and facts from topology and differential geometry. Here we present a completely combinatorial proof. The problem of finding such a proof was mentioned in [1].

Notice that $\mathcal{H}^{2n+1}$ is not an automatic group [2]. Therefore it is not hyperbolic, and so it does not have a subquadratic isoperimetric function ([3], [5]).

Theorem 1. *The 5-dimensional Heisenberg group* $\mathcal{H}^5 = \langle x, y, u, v | [x, u] = [x, v] = [y, u] = [y, v] = [x, y][v, u] = 1\rangle$ *has a quadratic isoperimetric function.*

This theorem and its proof can be easily extended to any Heisenberg group $\mathcal{H}^{2n+1}$ with $n \geq 2$ and some other 2-nilpotent groups which are central products.

The following lemma contains some well-known and obvious facts about $\mathcal{H}^5$. These facts will be used later without reference.

Lemma 2. 1. *$\mathcal{H}^5$ is nilpotent of class* 2.

2. *The derived subgroup $(\mathcal{H}^5)'$ is infinite cyclic.*

3. *The factor group $\mathcal{H}^5/(\mathcal{H}^5)'$ is free Abelian of rank* 4.

4. *For every two elements $a, b \in \mathcal{H}^5$ and any two integers m, n, we have $[a^m, b^n] = [a, b]^{mn}$.*

Definition 3. A word w is called a commutator word if it belongs to the derived subgroup of the free group. The notation $w(a, b)$ will mean that the word w depends on a, b only. If we substitute $a \to p$, $b \to q$ (where p, q are words), then the result of the substitution will be denoted by $w(p, q)$.

Lemma 4. *In $\mathcal{H}^5$, we have $(xu)^{y^{\pm 1}} = (xu)^{v^{\pm 1}}$, that is, $[xu, y^{\pm 1}v^{\mp 1}] = 1$.*
(We assume that $a^b = b^{-1}ab$, $[a, b] = a^{-1}b^{-1}ab$.)

Proof. $(xu)^y = x^y u = x[x, y]u = x[u, v]u = xu[u, v] = xu^v = (xu)^v$. This implies that $(xu)^{y^{-1}} = (xu)^{v^{-1}}$ since $[y, v] = 1$. □

The next lemma immediately follows from Lemma 4.

Lemma 5. *Every word of the form $y^{\pm 1}w(xu, v)y^{\mp 1}$ can be transformed into the word $v^{\pm 1}w(xu, v)v^{\mp 1}$ by using $O(|w|)$ applications of relations.*

Proof. $w(xu, v)^{y^{\pm 1}} = w((xu)^{y^{\pm 1}}, v^{y^{\pm 1}}) = w((xu)^{v^{\pm 1}}, v^{v^{\pm 1}}) = w(xu, v)^{v^{\pm 1}}$. □

Lemma 6. *Every commutator word $w(x, y)$ can be transformed into the word $w(u, v)$ by using at most $C_0|w|^2$ applications of defining relations of $\mathcal{H}^5$ for some $C_0 > 0$. In particular, $w(x, y) = w(u, v)$ in $\mathcal{H}^5$.*

Proof. Let n be the number of occurrences of the letter x in $w(x, y)$. Then the number of occurrences of x^{-1} is also n since w is a commutator word. Then we have

$$w(x, y) = w(x, y)u^n u^{-n} = w(xu, y).$$

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The last transformation requires $O(n|w|) = O(|w|^2)$ applications of commutation relations $[x, u] = [y, u] = 1$.

Our goal is to replace the letters $y^{\pm 1}$ in $w(xu, y)$ by $v^{\pm 1}$ by replacing subwords of the form $y^{\pm 1}p(xu, v)y^{\mp 1}$ by $v^{\pm 1}p(xu, v)v^{\mp 1}$ using Lemma 5 and the fact that the numbers of occurrences of y and y^{-1} in $w(x, y)$ coincide. Note that although $w(xu, y)$ does not contain v, this letter will appear in it after the first application of Lemma 5.

The number of such transformations is the number of occurrences of y in $w(x, y)$. Each transformation requires $O(|w|)$ relations, so the total number of relations needed is $O(|w|^2)$. □

Lemma 7. *Let $a, b \in \{x, x^{-1}, y, y^{-1}\}$. Then any word $w(a, b)$ of the form*

$$w_0 w_1 \ldots w_s \prod_{i=1}^{s} [a, b]^{q_i}$$

where w_i, q_i are words and $|q_i| \le Q$ for some number Q, can be transformed into the word

$$w_0 [x, y] w_1 [x, y] \ldots w_s$$

by using $O((|w| + Q)s)$ applications of relations.

Proof. First we transform in $\prod_{i=1}^{s}[a, b]^{q_i}$ each $[a, b]$ into $[u^{\pm 1}, v^{\pm 1}]$, then move the commutator number $i = 1, 2, \ldots$ to its spot between w_{i-1} and w_i (this requires $O(Q + |w|)$ commutativity relations for each i since $q_1, \ldots, q_{i-1}$ disappear when we start moving the ith commutator), and then transform each commutator $[u^{\pm 1}, v^{\pm 1}]$ back into $[a, b]$. □

Definition 8. For every integer m, n, ℓ, $0 \le \ell < m$, $m, n > 0$, let $R(m, n, \ell)$ be the following word $[x^m, y^n][x^\ell, y]^{y^n}$. If we draw this word on the plane R^2 (x-edges are horizontal, y-edges are vertical), then the image will be a rectangle with sides m and n with a horizontal ℓ by 1 rectangle on top. We define the *area* of the word $R(m, n, \ell)$ as $mn + \ell$. Note that by Lemma 2 $R(m, n, \ell) = [x, y]^{mn+\ell}$ in $\mathcal{H}^5$ and that the length of $R(m, n, \ell)$ in the free group is $2(m + n) + 2$.

The following statement is obvious.

Lemma 9. *For every integer m, n, ℓ, $0 < \ell \le m$, the word $R(m, n, 0)\prod_{i=1}^{\ell}[x, y]^{q_i}$, where the length of the words q_i does not exceed some number Q, can be transformed into $R(m, n, \ell)$ if $\ell < m$ or $R(m, n + 1, 0)$ if $\ell = m$. The number of applications of relations used in this transformation is $C_1(m + n + Q)\ell$ for some constant $C_1 > 0$.*

Proof. Indeed, it is enough to divide $R(m, n, 0)$ into $\ell + 1$ subwords $w_0, \ldots, w_\ell$ as

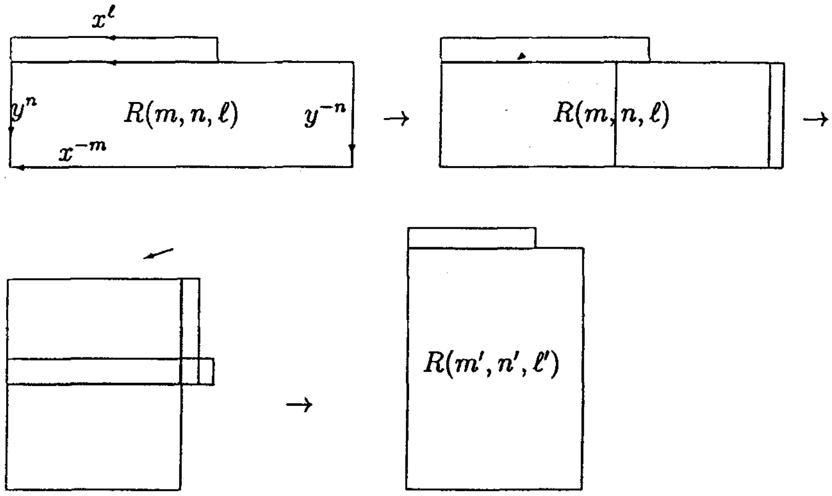
$$(x^{-m}y^{-n}x^{m-\ell+1}), x, x, \ldots, x, y^n$$

and use Lemma 7. The resulting word will be equal to $R(m, n, \ell)$ (if $\ell < m$) or $R(m, n + 1, 0)$ ($\ell = m$) in the free group. □

Lemma 10. *Let $a, b \in \{x, x^{-1}, y, y^{-1}\}$. Then in the free group, $[a^k, b] = \prod_{i=1}^{k}[a, b]^{a^{k-i}}$, $[a, b^k] = \prod_{i=0}^{k-1}[a, b]^{b^i}$.*

Lemma 11. *If $m > 3n$, then the word $R(m, n, \ell)$ can be transformed into a word $R(m', n', \ell')$ of the same area with $m' \le m/2$ and $n' \le 3m'$ by using $\le C_2 m^2$ defining relations for some constant $C_2 > 0$. In addition, $m' + n' \le m + n$.*

Proof



The idea of the proof is simple: we cut the rectangle approximately into two halves by a vertical line, then using Lemma 6 replace letters x and y in the right part by u and v, then move the right part on top of the left part using the commutativity relations, then replace u and v back into x and y, and finally distribute the small amount of extra squares to obtain a word of the form $R(m',n',\ell')$.

Let $m' = [m/2]$, $t = m - 2m'$. Note that $t = 0$ or 1. We have that

$R(m,n,\ell) =$ (definition)
$[x^m, y^n][x^\ell, y]^{y^n} =$ (in the free group)
$[x^{m-m'}, y^n]^{x^{m'}}[x^{m'}, y^n][x^\ell, y]^{y^n} =$ (by Lemma 6, $O((m+n)^2) = O(m^2)$ relations needed)
$[u^{m-m'}, v^n]^{x^{m'}}[x^{m'}, y^n][x^\ell, y]^{y^n} =$ ($O(m^2)$ commutativity relations)
$[x^{m'}, y^n][u^{m-m'}, v^n][x^\ell, y]^{y^n} =$ ($O(m^2)$ commutativity relations)
$[x^{m'}, y^n][u^{m-m'}, v^n]^{y^n}[x^\ell, y]^{y^n} =$ (by Lemma 6, $O(m^2)$ relations)
$[x^{m'}, y^n][x^{m-m'}, y^n]^{y^n}[x^\ell, y]^{y^n} =$ (in the free group)
$x^{-m'}y^{-n}x^{2m'-m}y^{-n}x^{m-m'}y^{2n}[x^\ell, y]^{y^n} =$ (in the free group)
$[x^{m'}, y^{2n}][y^{-n}, x^t]^{x^{m'}y^{2n}}[x^\ell, y]^{y^n}$

By Lemma 10, $[y^{-n}, x^t]$ is equal to $\prod_{i=1}^n [y^{-1}, x^t]^{y^{-(n-i)}}$ and $[y^{-1}, x^t] = [x^t, y]^{y^{-1}}$ in the free group. Again by Lemma 10, $[x^\ell, y] = \prod_{i=1}^\ell [x,y]^{x^{\ell-i}}$. Therefore

$$[x^{m'}, y^{2n}][y^{-n}, x^t]^{x^{m'}y^{2n}}[x^\ell, y]^{y^n} = [x^{m'}, y^{2n}]\prod_{i=1}^{s}[x,y]^{q_i} = R(m', 2n, 0)\prod_{i=1}^{s}[x,y]^{q_i}$$

where $|q_i| = O(m)$, $s \le n + \ell$. Note that

$$s \le n + \ell < m/3 + 2m' \le (2m'+1)/3 + 2m' < 3m'.$$

Since $s \le 3m'$, using Lemma 9 at most three times, one can transform the word $R(m', 2n, 0)\prod_{i=1}^{s}[x,y]^{q_i}$ into a word $R(m',n',\ell')$ for some n', ℓ' where $m'n' + \ell' = mn + \ell$ and $n' \le 2n + 2$.

It is clear that $m' \le m/2$ and $n' \le 3m'$ since $n' < m/3 + 2$. Finally $m' + n' \le m/2 + 2n + 2 \le m + n$: this is clearly true if $m \ge 10$ (since $m > 3n$) and can be easily checked for every $m < 10$ (for example, if $m = 7, n = 2, \ell = 6$, then $m' = 3$, $n' = 6$, $\ell' = 2$, and $m' + n' = m + n = 9$). □

The following lemma has exactly the same proof.

Lemma 12. *If $n > 3m$, then the word $R(m,n,\ell)$ can be transformed into a word $R(m',n',\ell')$ of the same area with $n' \le n/2$ and $m' \le 3n'$ by using $\le C_2n^2$ defining relations for some constant $C_2 > 0$. In addition, $m'+n' \le m+n$.*

We call the word $R(m,n,\ell)$ *thick* if both $m \le 3n$ and $n \le 3m$.

Lemma 13. *Every word $R(m,n,\ell)$ can be transformed into a thick word $R(m_0,n_0,\ell_0)$ of the same area by using at most $C_3(m+n)^2$ defining relations for $C_3 = \frac{4}{3}C_2$. In addition, $m_0+n_0 \le m+n$.*

Proof. Without loss of generality, assume that $m > 3n$. Then by Lemma 11 we can transform the word $R(m,n,\ell)$ into the word $R(m',n',\ell')$ of the same area with $m' \le m/2$, $n' \le 3m'$ by using at most C_2m^2 defining relations. If $m' > 3n'$, we can apply Lemma 11 again and get a word $R(m'',n'',l'')$ with $m'' \le m'/2 \le m/4$, $n'' \le 3m''$ by using at most $C_2(m')^2 \le C_2m^2/4$. Continuing in this manner, we finally get the desired thick word $R(m_0,n_0,\ell_0)$ by applying less than $C_2m^2(1+1/4+1/16+...) \le C_3m^2$ defining relations. The last statement of the lemma follows from Lemmas 11 and 12. □

Lemma 14. *Every two thick words $R_1 = R(m_1,n_1,\ell_1)$ and $R_2 = R(m_2,n_2,\ell_2)$ of the same area A can be transformed into one another by using at most C_4A defining relations for some constant $C_4 > 0$.*

Proof. We use induction on the area A. Note that if $m_1 = m_2$, then the words R_1 and R_2 are graphically equal since they have the same area $A = m_1n_1+\ell_1 = m_2n_2+\ell_2$, $\ell_1 < m_1, \ell_2 < m_2$.

So without loss of generality, assume that $m_1 > m_2$. Then $n_1 \le n_2$. Since both words R_1 and R_2 are thick, it is easy to check that each of the numbers m_1,n_1,m_2,n_2 is at least 1/3 of any other number in this list. This implies that each of these numbers is at most $2\sqrt{A}$.

We have that in the free group

$$R_1 = R(m_1-m_2,n_1,0)^{x^{m_2}}R(m_2,n_1,0)[x^{\ell_1},y]^{y^{n_1}}.$$

By Lemma 6 we can change the letters in $R(m_2,n_1,0)$ from x, y to u, v, commute the resulting word with $R(m_1-m_2,n_1,0)^{x^{m_2}}$, and then convert the letters u, v back to x and y by applying at most $8C_0(m_2+n_1)^2+4(m_2+n_1)(m_1+n_1) \le D_1A$ (for some constant D_1) defining relations. We get the word

$$R(m_2,n_1,0)R(m_1-m_2,n_1,0)^{x^{m_2}}[x^{\ell_1},y]^{y^{n_1}}.$$

Now applying Lemma 6 to $R(m_1-m_2,n_1,0)$, commuting with x^{m_2}, then replacing u,v with x,y (Lemma 6 again), we can remove the conjugation by x^{m_2} by using at most D_2A defining relations for some constant $D_2 > 0$. The word we get will have the form

$$R(m_2,n_1,0)R(m_1-m_2,n_1,0)[x^{\ell_1},y]^{y^{n_1}}.$$

Let $\ell_1 = (m_1-m_2)r+\ell'$, where $0 \le \ell' < m_1-m_2$. Applying Lemma 10 to $[x^{\ell_1},y]$ and then applying Lemma 9 r or $r+1$ times (depending on whether $\ell' = 0$ or not), one can transform the word $R(m_1-m_2,n_1,0)[x^{\ell_1},y]^{y^{n_1}}$ into $R(m_1-m_2,n',\ell')$, where $n' = n_1+r < n_1+m_1$, by using at most

$$C_1(m_1-m_2+n_1+m_1+n_1)((m_1-m_2)r+\ell') = C_1(m_1-m_2+n_1+m_1+n_1)\ell_1\ leD_3A$$

applications of defining relations. The resulting word has the form

$$R(m_2,n_1,0)R(m_1-m_2,n',\ell').$$

Now by Lemma 13 we can transform the second factor of this product into a thick word $R(m_3,n_3,\ell_3)$ with the same area and $m_3+n_3 \le m_1-m_2+n'$ by using $C_3(m_1-m_2+n')^2 \le D_4A$ defining relations for some constant D_4.

The area of $R(m_3, n_3, \ell_3)$ is $A - m_2 n_1 = m_1 n_1 + \ell_1 - m_2 n_1 < \frac{5}{6}A$ since $m_2 \geq \frac{1}{3}m_1$ and $\ell_1 < m_1 n_1$.

Now let us deal with $R(m_2, n_2, \ell_2)$. We can rewrite it in the free group as follows:

$$R_2 = R(m_2, n_1, 0) R(m_2, n_2 - n_1, \ell_2)^{y^{n_1}}$$

if $n_2 > n_1$, or $R_2 = R(m_2, n_1, 0)$ if $n_1 = n_2$.

By using Lemma 6 we can remove (as before) the conjugation by y^{n_1} using at most $D_2 A$ defining relations. After that we can transform the word $R(m_2, n_2 - n_1, \ell_2)$ into a thick word $R(m_4, n_4, \ell_4)$ of the same area by using at most $D_4 A$ defining relations.

The area of $R(m_4, n_4, \ell_4)$ is equal to $A - m_2 n_1$, so it is equal to the area of $R(m_3, n_3, \ell_3)$ and is at most $\frac{5}{6}A$.

Now let us denote the constant $6(D_1 + 2D_2 + D_3 + 2D_4)$ by C_4 and assume by induction that every two thick words of area $B < A$ can be transformed into one another by applying at most $C_4 B$ defining relations.

Then we can transform the word $R(m_3, n_3, \ell_3)$ into $R(m_4, n_4, \ell_4)$ by using at most $\frac{5}{6}C_4 A$ defining relations. This implies that we can transform R_1 into R_2 by applying at most $D_1 A + D_2 A + D_3 A + D_4 A + D_2 A + D_4 A + \frac{5}{6}C_4 A = C_4 A$ defining relations.□

Lemma 15. *For any two words $R(m_1, n_1, \ell_1)$ and $R(m_2, n_2, \ell_2)$ of the same area, one can transform one word into another by using at most $C_5(m_1 + n_1 + m_2 + n_2)^2$ defining relations for some constant $C_5 > 0$.*

Proof. This follows immediately from Lemmas 13 and 14: first we transform both words into thick words of the same area, then transform one thick word into another; we use the fact that when we transform a word $R(m, n, \ell)$ into a thick word $R(m', n', \ell')$ then $m' + n' \leq m + n$ (Lemma 13) and the trivial observation that the area of a word $R(m, n, \ell)$ does not exceed $(m + n)^2$. □

Lemma 16. *For every two words $R_1 = R(m_1, n_1, \ell_1)$ and $R_2 = R(m_2, n_2, \ell_2)$ of areas A_1 and A_2, their product $R_1 R_2$ can be transformed into a thick word $R(m_3, n_3, \ell_3)$ with area $A_1 + A_2$ by using at most $C_6(m_1 + n_1 + m_2 + n_2)^2$ defining relations for some constant $C_6 > 0$.*

Proof. We shall consider only the case where $A_1 \geq A_2$. The other case is similar. By Lemma 15 we can transform R_2 into a word R' where either $R' = R(m_1, n', \ell')$ (if $A_2 \geq m_1$) or $R' = R(A_2, 1, 0)$ (if $A_2 < m_1$) by using at most $O((m_1 + n_1 + m_2 + n_2)^2)$ defining relations.

As before, we consider only the most difficult case where $A_2 \geq m_1$ (in the other case the number of transformations will be smaller). We have

$$R_1 R' = R(m_1, n_1, 0)[x^{\ell_1}, y]^{y^{n_1}} R(m_1, n', 0)[x^{\ell'}, y]^{y^{n'}}.$$

Using Lemma 6 we can permute $[x^{\ell_1}, y]^{y^{n_1}}$ and $R(m_1, n', 0)$, and replace $R(m_1, n', 0)$ by $R(m_1, n', 0)^{y^{n_1}}$ (we lift the second rectangle onto the top of the first rectangle). Note that $R(m_1, n_1, 0) R(m_1, n', 0)^{y^{n_1}} = R(m_1, n_1 + n', 0)$ in the free group. Then, using Lemmas 9 and 10, we convert the word

$$R(m_1, n_1 + n', 0)[x^{\ell_1}, y]^{y^{n_1}}[x^{\ell'}, y]^{y^{n'}}$$

into the word $R(m_1, n_3, \ell_3)$ for some n_3 and ℓ_3. Now we can make this word thick by using Lemma 13. All these transformations require application of $O((m_1 + n_1 + m_2 + n_2)^2)$ defining relations. □

Lemma 17. *For every two words $R_1 = R(m_1, n_1, \ell_1)$ and $R_2 = R(m_2, n_2, \ell_2)$ of areas A_1 and A_2, their product $R_1 R_2{}^{-1}$ can be transformed into an empty word or a word $R^{\pm 1} = R(m_3, n_3, \ell_3)^{\pm 1}$ such that R is thick and the area of R is $|A_1 - A_2|$ by using at most $C_7(m_1 + n_1 + m_2 + n_2)^2$ defining relations for some constant $C_7 \geq C_6$.*

Proof. If $A_1 = A_2$, then we apply Lemma 15. So let $A_1 \neq A_2$. Without loss of generality assume that $A_1 > A_2$. Consider a thick word $R_3 = R(m_3, n_3, \ell_3)$ of area $A_1 - A_2$. Then $m_3 + n_3 = O(m_1 + n_1 + m_2 + n_2)$ (since R_3 is thick). By Lemma 16, we can transform R_3R_2 into a thick word $R' = R(m', n', \ell')$ with area A_1 and $m' + n' = O(m_1 + n_1 + m_2 + n_2)$. By Lemma 15, we can transform R' into R_1. The total number of applications of defining relations in all these transformations is $O((m_1 + n_1 + m_2 + n_2)^2)$. Thus, one can transform $R_1R_2^{-1}$ into $R_3R_2R_2^{-1} = R_3$ by using $O((m_1 + n_1 + m_2 + n_2)^2)$ defining relations. □

Lemma 18. *Every commutator word of the form $W = x^a w(x,y) y^b$ can be transformed either into the empty word or into a word $R(m,n,\ell)^{\pm 1}$ for some m, n, ℓ such that $R(m,n,\ell)$ is a thick word and its area, $mn + \ell$, is at most $|w|^2$ by using $C_8|w|^2$ defining relations, where $C_8 = 250C_0 + 70C_7$.*

Proof. Induction on the length of w. The base ($|w| = 1$) is obvious.

Let us write w as w_1w_2, where $|w_1| = [|w|/2]$. Let p (resp. q) be the sum of exponents of x (resp. y) in w_1. Then in the free group, we have

$$W = (x^{a+p}x^{-p}w_1y^{-q}x^{-(a+p)})(x^{a+p}y^qx^{-(a+p)}y^{-q})(y^qx^{a+p}w_2y^{b+q}y^{-q}).$$

Note that $|a+p|$ does not exceed the number of occurrences of x in w_2 and $|q|$ does not exceed the number of occurrences of y in w_1. Therefore $|a+p| + |q| \leq |w|$.

Note that $x^{-p}w_1y^{-q}$ and $x^{a+p}w_2y^{b+q}$ are commutator words by the definition of p and q. By the induction hypothesis these words can be converted to the empty word or words of the form $R_i^{\pm 1} = R(s_i, t_i, \ell_i)^{\pm 1}$ for some s_i, t_i, ℓ_i, $i = 1, 2$, where R_i is a thick word of area at most $|w_i|^2$. The number of defining relations needed for these transformations is $C_8(|w_1|^2 + |w_2|^2) \leq \frac{5}{9}C_8|w|^2$ (the worst case here is when $|w| = 3, |w_1| = 1, |w_2| = 2$).

Since R_i, $i = 1, 2$, is a thick word, $s_i + t_i \leq 3|w_i|$ (indeed, $3s_i \geq t_i, 3t_i \geq s_i$, and $s_it_i \leq |w_i|^2$).

Therefore, by using Lemma 6 we can remove conjugations by x^{a+p} and y^{-q}. The number of defining relations needed for these transformation is at most $100C_0|w|^2$ (the length of R_i is at most $2 \cdot 3|w_i| + 2 \leq 8|w_i|$; we need to replace x, y by u, v, then commute resulting words with the conjugating elements, then convert x, y back to u, v).

The commutator $x^{a+p}y^qx^{-(a+p)}y^{-q}$ is conjugated in the free group to a word of the form $R_3^{\pm 1}R(|a+p|, |q|, 0)^{\pm 1}$, and the conjugating word has length at most $|w|$. This conjugating word can be removed by using Lemma 6, which takes at most $10C_0|w|^2$ defining relations. The area of R_3 is $|a+p||q| \leq |w|^2/4$ since $|a+p| + |q| \leq |w|$.

The resulting word is a product of three words $R_1^{\pm 1}R_3^{\pm 1}R_2^{\pm 1}$. Applying Lemmas 16 and 17, we can combine these words into a word $R(m,n,\ell)^{\pm 1}$. This operation takes at most $30C_7|w|^2$.

The area of $R(m,n,\ell)$ is at most the sum of the areas of R_1, R_2, and R_3 (by Lemmas 16 and 17). Therefore, the area of $R(m,n,\ell)$ is at most $\frac{5}{9}|w|^2 + \frac{1}{4}|w|^2 \leq |w|^2$ (as desired).

The total number of defining relations needed in order to get $R(m,n,\ell)^{\pm 1}$ is at most

$$(\frac{5}{9}C_8 + 110C_0 + 30C_7)|w|^2 \leq C_8|w|^2. \quad \square$$

Proof of the Theorem. Let $w = w(x,y,u,v)$ be any word in x,y,u,v which is equal to 1 in $\mathcal{H}^5$. Using $O(|w|^2)$ commutativity relations, we can transform w into a word of the form $w_1(x,y)w_2(u,v)$ which has the same length as w. By part 3 of Lemma 2, both w_1 and w_2 are commutator words. Therefore, by Lemma 6 we can transform $w_2(u,v)$ into $w_2(x,y)$ by using $O(|w|^2)$ defining relations. Thus we can assume that $w = w(x,y)$.

By Lemma 18, for $a = b = 0$ we can transform the word w into a word $R^{\pm 1} = R(m,n,\ell)^{\pm 1}$ or the empty word by using $O(|w|^2)$ defining relations. By Lemma 2, the word R is equal to $[x,y]^{mn+\ell} \neq 1$ in $\mathcal{H}^5$ because $[x,y]$ is an element of infinite order and $m, n > 0, \ell \geq 0$. Therefore, our transformation changes w into the empty word. Therefore, w can be transformed into the empty word by using $O(|w|^2)$ defining relations. Thus, $\mathcal{H}^5$ has a quadratic isoperimetric function. $\square$

Remark. The method used in this paper can be employed to find isoperimetric functions of other nilpotent groups. For example, one can prove that the central product of $l \geq 2$ copies of any finitely generated free nilpotent group of class 2 with derived subgroups identified in the natural way has a quadratic isoperimetric function. We can also prove that such a central product of two copies of any finitely generated nilpotent group of class 2 has an isoperimetric function equivalent to $n^2 \log n$. So far, we have been unable to prove that these groups always have quadratic isoperimetric functions.

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