On subgroups of R. Thompson's group F and other diagram groups

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Abstract

In this paper, we continue our study of the class of diagram groups. Simply speaking, a diagram is a labelled plane graph bounded by a pair of paths (the top path and the bottom path). To multiply two diagrams, one simply identifies the top path of one diagram with the bottom path of the other diagram, and removes pairs of "reducible" cells. Each diagram group is determined by an alphabet X, containing all possible labels of edges, a set of relations $\mathcal{R} = \{ u_i = v_i \mid i = 1, 2, \dots \}$, containing all possible labels of cells, and a word w over X – the label of the top and bottom paths of diagrams. Diagrams can be considered as 2-dimensional words, and diagram groups can be considered as 2-dimensional analogue of free groups. In our previous paper, we showed that the class of diagram groups contains many interesting groups including the famous R. Thompson group F (it corresponds to the simplest set of relations $\{ x = x^2 \}$), closed under direct and free products and some other constructions. In this paper we study mainly subgroups of diagram groups. We show that not every subgroup of a diagram group is itself a diagram group (this answers a question from the previous paper). We prove that every nilpotent subgroup of a diagram group is abelian, every abelian subgroup is free, but even the Thompson group contains solvable subgroups of any degree. We also study distortion of subgroups in diagram groups, including the Thompson group. It turnes out that centralizers of elements and abelian subgroups are always undistorted, but the Thompson group contains distorted soluble subgroups.

Introduction

This paper is devoted to further study of the so called *diagram groups*. The definition of diagram groups was first given by Meakin and Sapir in 1995. Their student Vesna Kilibarda obtained first results about diagram groups in her thesis [17] (see also her paper [18]). Further results about diagram groups have been obtained in our paper [12]. Here we survey the main results of that paper (see [12] for details).

Diagram groups reflect certain important properties of semigroup presentations. For instance we showed that three definitions of asphericity given by Pride [27] are in fact equivalent and are equivalent to the triviality of all diagram groups over the presentation. One can say that diagram groups measure the non-asphericity of semigroup presentations.

On the other hand, it turned out that the class of diagram groups is interesting even if we forget about its connection with semigroup presentations. These groups have nice algorithmic properties: the word problem in every diagram group is solvable in time $O(n^{2+\varepsilon})$ for every $\varepsilon > 0$. This does not depend on whether the word problem for the corresponding semigroup presentation is solvable or not. If the word problem of this semigroup presentation is solvable then the conjugacy problem is solvable in the corresponding diagram group.

If a group is representable by diagrams (i.e. it is a subgroup of a diagram group) then one can use geometry of planar graphs to deduce certain properties of the group. Diagrams can be viewed as "2-dimensional words" and in [12], we developed a calculus called "combinatorics on diagrams", which is parallel to the well known combinatorics on words (see Lothaire [21]).

Geometry of diagrams allows one to consider many homomorphism from diagram groups into the group of piecewise linear homeomorphisms of the real line. Thus we have a connection between groups representable by diagrams and groups representable by piecewise linear functions. This connection can be used in both directions.

We showed that the class of diagram groups is wide. It contains the free groups, free abelian groups, the R. Thompson group F and its generalizations found by Brown [4]. This class is closed under finite direct products, arbitrary free products and some other constructions. Note that the Thompson group is the diagram group over the following simple presentation $\langle\, x \mid x^2 = x \,\rangle$. In [12] we obtained several previously unknown results about Thompson's group, essentially using its representation as a diagram group.

- The conjugacy problem in F is solvable.
- The centralizer of each element of F is a finite direct product of groups each of which is either a copy of F or an infinite cyclic group $\mathbf{Z}$.

Let us give a short summary of the content of this paper.

Section 1 contains the list of the main concepts used in this paper.

In Section 2, we introduce the concept of diagram product of groups. It is defined as the fundamental group of a certain 2-complex of groups. Theorem 4, the main result of this section, states that the class of diagram groups is closed under diagram products. It turns out that all "products" considered before (the free product, the direct product, etc.) are particular cases of the diagram product. Examples 5 – 8, 10, 12 show applications of Theorem 4. In partricular we prove that the class of diagram groups is closed under countable direct powers (Theorem 9), wreath products with $\mathbf{Z}$ (Theorem 11), and a certain special construction $\mathcal{O}(G,H)$ (Theorem 13), whose role will be clear later.

In Section 3, we show that nilpotent subgroups of diagram groups are abelian (Corollary 15), and abelian subgroups are free abelian (Theorem 16). We finish this section with a description of sets of pairwise commuting diagrams (Theorem 17).

In Section 4, we prove that the Thompson group F contains subgroups isomorphic to the restricted wreath product of two infinite cyclic groups and soluble subgroups of arbitrary degree (this was first proved by Brin). It turns out that for any subgroup of piecewise linear functions (including F) there exists a dichotomy: either it contains **Z** wr **Z** or it is abelian (Theorem 21). This implies, in particular, that a non-abelian subgroup of the group of piecewise linear functions cannot be a one-relator group (Corollary 23). This result strengthens the well known fact that the group of piecewise linear functions does not contain free non-abelian subgroups.

In Section 4, we also give necessary and sufficient conditions for a diagram group to contain a copy of **Z** wr **Z** as a subgroup (Theorem 24). We study the question when a diagram group over some semigroup presentation $\mathcal{P}$ contains a copy of the Thompson group F. We prove that if the semigroup given by $\mathcal{P}$ contains an idempotent then a diagram group over this presentation contains a copy of F. (Theorem 25). The interesting question of whether the converse statement holds is open.

In Section 5, we present a counterexample to the Subgroup Conjecture. This conjecture stated that every subgroup of a diagram group is a diagram group itself. It was motivated by the similarity between diagram groups and free groups. At first we thought that the conjecture is easy to disprove and the derived subgroup F' of F is a counterexample. But it turned out that F' is a diagram group (Theorem 26). This solves several problems from [12]. We asked whether a diagram group can coincide with its derived subgroup and whether every diagram group has an LOG-presentation. Corollary 27 gives a positive answer to the first question and a negative answer to the second question.

But the main result of this Section is Theorem 28 which shows that the one-relator group $\langle\, x, y \mid xy^2x = yx^2y\,\rangle$ is not a diagram group but is isomorphic to a subgroup of a diagram group. In the proof, we use the construction $\mathcal{O}(G,H)$ from Section 2. This gives a counterexample to the Subgroup Conjecture. Nevertheless, we think that in many partricular cases this conjecture is true and we pose several open question in this regard.

At the end of this section we study the following series of groups

$$G_n = \langle\, x_1, \ldots, x_n \mid [x_1, x_2] = [x_2, x_3] = \cdots = [x_{n-1}, x_n] = [x_n, x_1] = 1\,\rangle.$$

For $n \leq 4$ these groups are diagram groups; we prove (Theorem 30), that for odd $n \geq 5$ this is not so. It is not known whether these groups are representable by diagrams. If so, this will give us new counterexamples to the Subgroup Conjecture.

The last Section 6 is devoted to the distortrion of subgroups in diagram groups. In a recent paper [5] Burillo prroved that for every natural n the Thompson group F contains subgroups isomorphic to $F \times \mathbf{Z}^n$ and quasi-isometrically (without distortrion) embedded into F. A similar fact is true for an embedding of $F \times F$. We prove (Theorem 34) that every centralizer of an element in F is embedded into F without distortion. Centralizers of elements of F can be arbitrary finite direct products of copies of F and copies of **Z**. Burillo also proved that every cyclic subgroup of F is embedded without distortion (this fact is also an immediate corollary of Lemma 15.29 in [12]). We prove (Theorem 33) that not only cyclic but arbitrary finitely generated abelian subgroups of **any** diagram groups are undistorted. Finally we found solvable subgroups of F which are distorted.

Theorem 38 shows that for every natural $d \geq 2$ there exists a finitely generated solvable subgroup K_d in F such that its distortion function is at least n^d.

Acknowledgements. The authors thank M. Brin and S. Pride for helpful discussions of the results of this paper.

1 Preliminaries

For an alphabet Σ let Σ^+ denote the free semigroup over Σ, and let Σ^* denote the free monoid. Elements of the free monoid are called *words*. The identity element, i.e. the empty word, is denoted by 1.

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a presentation of a semigroup where Σ is an alphabet, $\mathcal{R}$ is a set of pairs of non-empty words over Σ. The semigroup S given by $\mathcal{P}$ is the factor-semigroup $\Sigma^+/\sim$ where $\sim$ is the smallest congruence on Σ^+ containing $\mathcal{R}$. Elements of Σ are called *generators*, pairs $(u,v) \in \mathcal{R}$ written also as $u = v$ are called *defining relations*. Left and right parts of defining relations are called *defining words*. We shall assume that all presentations are anti-symmetric that is if $u = v \in \mathcal{R}$ then $v = u \notin \mathcal{R}$. In particular $\mathcal{R}$ does not contain relations $u = u$.

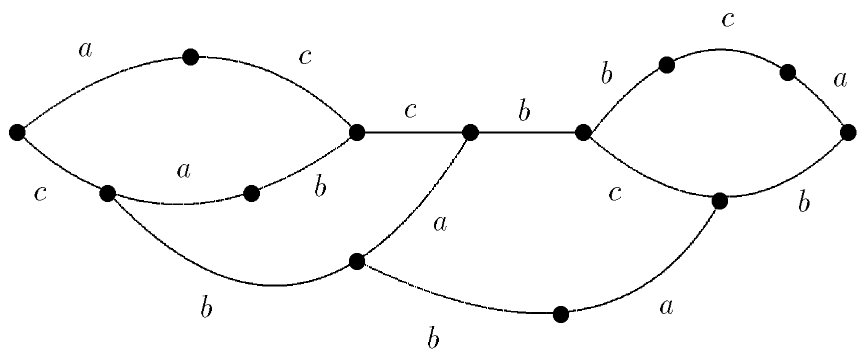
With any semigroup presentation $\mathcal{P}$, we associate the following graph $\Gamma(\mathcal{P})$. The vertices are all words in Σ^+. Edges are the elements of $\Sigma^* \times \mathcal{R}^{\pm 1} \times \Sigma^*$. We shall denote edges by $(x, u \to v, y)$, where $x, y \in \Sigma^*$, and either $(u,v) \in \mathcal{R}$ or $(v,u) \in \mathcal{R}$. If $e = (x, u \to v, y)$ then the *inverse edge* is defined by $e^{-1} = (x, v \to u, y)$. The *initial vertex* of e is $\iota(e) = xuy$ and the *terminal vertex* is $\tau(e) = xvy$. Thus vertices of this graph are words and edges are elementary transformations of words (i.e. substitutions of defining words by their pairs). The graph $\Gamma(\mathcal{P})$ describes all derivations over $\mathcal{P}$: two non-empty words w_1, w_2 are *equal modulo* $\mathcal{P}$ if and only if there exists a path in the graph connecting w_1 and w_2. This path is called a *derivation* of w_2 from w_1.

With every derivation over $\mathcal{P}$, one can associate a geometric object, a *semigroup diagram* over $\mathcal{P}$. Semigroup diagrams were first introduced by E. V. Kashintsev [16] and then rediscovered by Remmers [28] and others(see [15, 32]). We do not give an exact definition here (see also [12]), the definition will be clear from the following example.

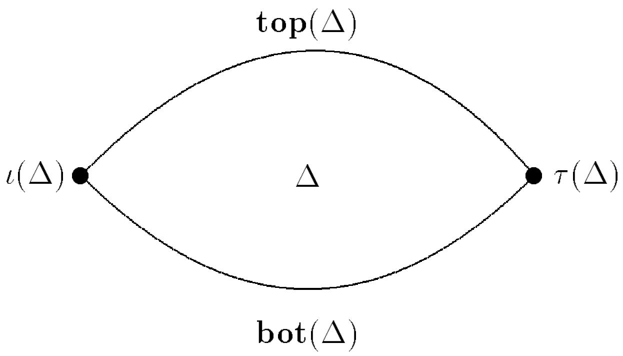
Example 1 Let $\mathcal{P} = \langle a, b, c \mid abc = ba, bca = cb, cab = ac \rangle$. Consider the following derivation over $\mathcal{P}$:

$$(1, ac \to cab, cb^2ca)(c, abc \to ba, b^2ca)(cbab, bca \to cb, 1)(cb, abc \to ba, b).$$

The corresponding diagram Δ over $\mathcal{P}$ is the following:



Let us introduce the terminology associated with diagrams. Every diagram over $\mathcal{P}$ is a planar graph. It has *vertices*, *edges* and *cells*. In Example 1, the diagram Δ has 13 vertices, 16 edges and 4 cells. The number of cells is equal to the length of the corresponding derivation. Each positive edge has a *label* from Σ, positive edges are oriented from left to right. The label of an edge e is denoted by $\varphi(e)$. We shall consider only positive paths in Δ, that is paths consisting of positive edges. For every path p in a diagram Δ, its *label* $\varphi(p)$ is the word read on the path. Any diagram Δ has the *initial verrtex* $\iota(\Delta)$ and the *terminal vertex* $\tau(\Delta)$, the *top path* $\mathbf{top}(\Delta)$ and the *bottom path* $\mathbf{bot}(\Delta)$ connecting the initial and terminal vertices. The diagram Δ lies between its top and bottom paths. This notation is illustrated by the following example.

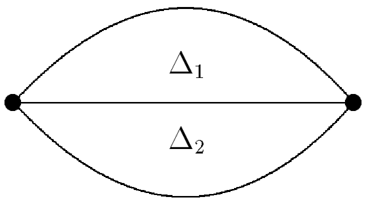


Notice that paths $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$ can have common edges. Every cell π of a diagram is a diagram itself, so we can define the notation $\iota(\pi)$, $\tau(\pi)$, $\mathbf{top}(\pi)$, $\mathbf{bot}(\pi)$ and the corresponding concepts. If words u and v are labels of the top and the bottom paths of a cell π then either $u = v$ or $v = u$ is a defining relation (that is it belongs to $\mathcal{R}$). In this case we call π a (u, v)-cell.

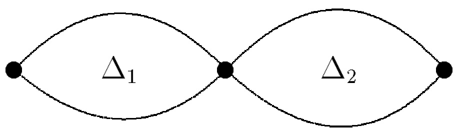
For every non-empty word w, there exists a *trivial diagram* $\varepsilon(w)$ without cells whose top and bottom paths coincide and have label w.

We do not distinguish isotopic diagrams. The notation $\Delta_1 \equiv \Delta_2$ means that Δ_1 and Δ_2 are isotopic.

If the label of $\mathbf{top}(\Delta)$ is w_1 and the label of $\mathbf{bot}(\Delta)$ is w_2 then Δ is called a (w_1, w_2)-diagram. Let w_1, w_2, w_3 be any three vertices of the graph $\Gamma(\mathcal{P})$ and let p_i $(i = 1, 2)$ be paths in the graph $\Gamma(\mathcal{P})$ from w_i to w_{i+1}. By Δ_i, we denote the diagram corresponding to the path p_i $(i = 1, 2)$. It is easy to see that the product of paths p_1 and p_2 in the graph corresponds to the diagram Δ obtained from Δ_1 and Δ_2 by identifying the bottom path of Δ_1 and the top path of Δ_2. The resulting diagram Δ will be called the *composition* of diagrams Δ_1 and Δ_2, we denote it by $\Delta_1 \circ \Delta_2$. Thus $\circ$ is a partial operation on the set of all diagrams over $\mathcal{P}$. The composition of a (w_1, w_2)-diagram and a (w_2, w_3)-diagram is a (w_1, w_3)-diagram. For every word $w \in \Sigma^+$, the set of all (w, w)-diagrams over $\mathcal{P}$ is a semigroup with respect to the operation $\circ$. Diagrams of this form will be called *spherical* diagrams with *base* w. This semigroup has the identity element $\varepsilon(w)$. We define also another associative operation on the set of all diagrams over $\mathcal{P}$. Namely the *sum* $\Delta_1 + \Delta_2$ of diagrams Δ_1 and Δ_2 is the diagram obtained by identifying $\tau(\Delta_1)$ and $\iota(\Delta_2)$. These two operations are illustrated by the following figure:



$\Delta_1 \circ \Delta_2$



$\Delta_1 + \Delta_2$

Suppose that a diagram Δ contains a (u, v)-cell and a (v, u)-cell such that the top path of the first cell is the bottom path of the second cell. Then we say that these two cells form a *dipole*. In this case we can remove these two cells by first removing their common path, and then identifying the bottom path of the first cell with the top path of the second cell. A diagram is called *reduced* if it does not contain dipoles. One can get a reduced diagram from any diagram by removing dipoles. Kilibarda [17] proved that every diagram has a unique reduced form. We call two diagrams Δ_1 and Δ_2 *equivalent*, written as $\Delta_1 \cong \Delta_2$, if their reduced forms are the same. It is easy to see that if $\Delta_1 \cong \Delta_2$, $\Delta_3 \cong \Delta_4$ then $\Delta_1 \circ \Delta_3 \cong \Delta_2 \circ \Delta_4$ and $\Delta_1 + \Delta_3 \cong \Delta_2 + \Delta_4$.

Therefore on the set $\mathcal{D}(\mathcal{P}, w)$ of all equivalence classes of (w, w)-diagrams one can define a product, by setting $[\Delta_1] \cdot [\Delta_2] = [\Delta_1 \circ \Delta_2]$, where square brackets denote equivalence classes.

The product of a (w_1, w_2)-diagram Δ and the (w_2, w_1)-diagram Δ', which is a mirror image of Δ is obviously equivalent to the trivial diagram $\varepsilon(w_1)$. The diagram Δ' will be denoted by Δ^{-1}.

For simplicity we shall call equivalent diagrams *equal*, use "$=$" instead of "$\cong$", and drop square brackets and the multiplication sign. So for every $\Delta \in \mathcal{D}(\mathcal{P}, w)$ $\Delta\Delta^{-1} = \varepsilon(w)$. As a result $\mathcal{D}(\mathcal{P}, w)$ turns out to be a group which is called the *diagram group* over the semigroup presentation $\mathcal{P}$ with base w. Since every equivalence class contains a unique reduced diagram, one can assume that $\mathcal{D}(\mathcal{P}, w)$ consists of reduced diagrams with the natural multiplication ($\Delta_1\Delta_2$ is the reduced form of $\Delta_1 \circ \Delta_2$).

In what follows, the term *diagram group* means a diagram group over some presentation with some base.

We shall use the standard notation for conjugation in groups: $a^b = b^{-1}ab$, and for the commutator: $[a, b] = a^{-1}a^b = a^{-1}b^{-1}ab$. If A and B are subgroups of a group G then $[A, B]$ denotes the subgroup generated by all commutators $[a, b]$ where $a \in A$, $b \in B$.

Now let us shortly describe some results about diagram groups obtained earlier.

The diagram group corresponding to the presentation $\mathcal{P} = \langle\, x \mid xx = x \,\rangle$ with base x is the famous R. Thompson's group F, which has the following presentation:

$$\langle\, x_0, x_1, \ldots \mid x_j^{x_i} = x_{j+1}\ (j > i) \,\rangle.$$

(see [12, Example 6.4].

This group has several interesting propertries and is studied by mathematicians working in different areas of mathematics (λ-calculus, functional analysis, homological algebra, homotopy theory, group theory). It was discovered by R. Thompson in 1965, and was rediscovered later by other authors. [6] presents a survey of results about F. Since F is one of the most important diagram groups, and since we are going to present some new results about it in this paper, let us recall some known properties of this group. These propertries can be found in [6], [12] and [11].

1. The group F is isomorphic to the group of all increasing continuous piecewise linear maps of the interval $[0, 1]$ onto itself such that the singularities occur at finitely many dyadic points (points of the form $m/2^n$) and all slopes are powers of 2. The group operation is the composition of functions (we shall write function symbols to the right of the argument).

2. In the previous paragraph, one can replace the interval $[0, 1]$ by $[0, +\infty]$, adding the assumption that the slop on $+\infty$ is 1. The resulting group is also isomorphic to F.

3. F does not satisfy any non-trivial identity.

4. F does not contain any free non-abelian subgroups. Every subgroup of F either is abelian or contains an infinite direct power of $\mathbf{Z}$.

5. F is finitely presented, it has a presentation with two generators and two defining relations. The word problem and the conjugacy problem are solvable in F. It has a polynomial isoperimetric function [11].

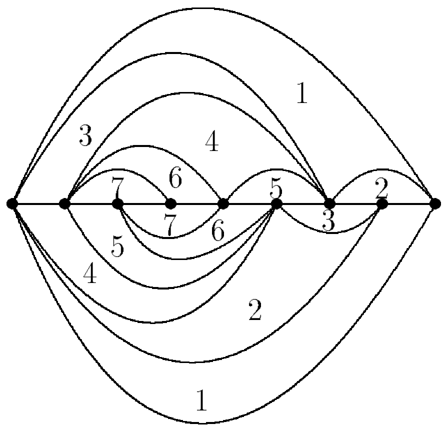
There exists a clear connection between representation of elements of F by diagrams and normal form of elements in F. Recall [6] that every element in F is uniquely representable in the following form:

$$x_{i_1}^{s_1} \ldots x_{i_m}^{s_m} x_{j_n}^{-t_n} \ldots x_{j_1}^{-t_1}, \tag{1}$$

where $i_1 \le \cdots \le i_m \ne j_n \ge \cdots \ge j_1$; $s_1, \ldots, s_m, t_1, \ldots t_n \ge 0$, and if x_i and x_i^{-1} occur in (1) for some $i \ge 0$ then either x_{i+1} or x_{i+1}^{-1} also occurs in (1). This form is called the *normal form* of elements in F. (Note that in [13], we constructed another normal form for elements of F, our normal forms are locally testable.)

Let us show how given an (x,x)-diagram over $\mathcal{P} = \langle\, x \mid xx = x \,\rangle$ one can get the normal form of the element represented by this diagram. We simply describe the procedure providing no proofs. Details can be deduced from [12, Example 6.4].

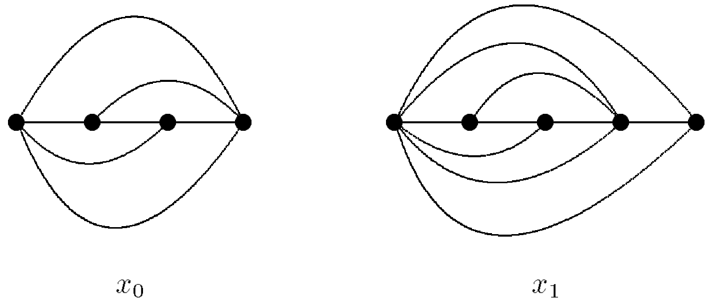
Example 2 Every diagram Δ over $\mathcal{P}$ can be divided by its longest positive path from its intitial vertex to its terminal vertex into two parts, *positive* and *negative*, denoted by Δ^+ and Δ^-, respectively. So $\Delta = \Delta^+ \circ \Delta^-$. It is easy to prove by induction on the number of cells that all cells in Δ^+ are (x, x^2)-cells, all cells in Δ^- are (x^2, x)-cells. This implies that the numbers of cells in Δ^+ and in Δ^- are the same. Denote this number by k. Let us number the cells of Δ^+ by numbers from 1 to k by taking every time the "rightmost" cell, that is, the cell which is to the right of any other cell attached to the bottom path of the diagram formed by the previous cells. The first cell is attached to the top path of Δ^+ ($= \mathbf{top}(\Delta)$). The ith cell in this sequence of cells corresponds to an edge of the graph $\Gamma(\mathcal{P})$, which has the form $(x^{\ell_i}, x \to x^2, x^{r_i})$, where ℓ_i (r_i) is the length of the path from the initial (resp. terminal) vertex of the diagram (resp. the cell) to the initial (resp. terminal) vertex of the cell (resp. the diagram), and contained in the bottom path of the diagram formed by the first $i-1$ cells. If $\ell_i = 0$ then we label this cell by 1. If $\ell_i \ne 0$ then we label this cell by the element x_{r_i} of F. Multiplying the labells of all cells, we get the "positive" part of the normal form. For example, the diagram on the next picture



the positive part is equal to $x_0 x_2^2 x_4 x_5$ (cells 1 and 3 were labelled by the identity element).

In order to find the "negative" part of the normal form, consider $(\Delta^-)^{-1}$, number its cells as above and label them as above. In our example, we get the word $x_1 x_3^2 x_4$ (cells 1, 2, 4 are labelled by 1). Thus the "negative" part of the normal form is $(x_1 x_3^2 x_4)^{-1}$, and it remains to multiply the positive and negative parts. In our example, the normal form is $x_0 x_2^2 x_4 x_5 x_4^{-1} x_3^{-2} x_1^{-1}$.

Diagrams presented below are generators x_0, x_1 of the group F. They generate the whole F.



One can ask several natural general questions about diagram groups. Which groups are diagram groups? Which groups are representable by diagrams (are subgroups of diagram groups)? Do these two classes coinside? How to compute a diagram group over a given presentation $\mathcal{P}$ and with a given base?

There exists a well developed technology for computing diagram groups. The starting point for computing diagram groups is Kilibarda's theorem about fundamental groups of Squier complexes. In order to formulate this important result, let us define the structure of a 2-complex on the graph $\Gamma(\mathcal{P})$ for any presentation $\mathcal{P}$.

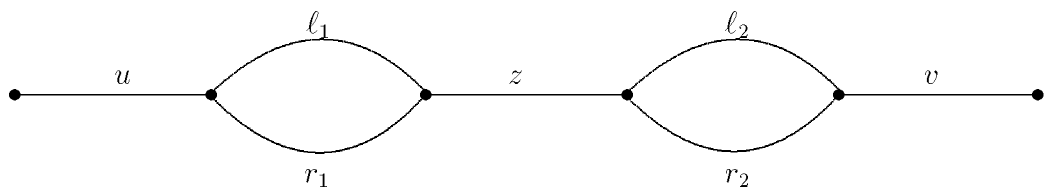
First notice that although for every path in $\Gamma(\mathcal{P})$ there exists a unique diagram associated with this path, the same diagram can be associated with many paths. Consider the following typical case:

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ where $\mathcal{R}$ contains two defining relations $\ell_i = r_i$ $(i = 1, 2)$, and let u, v, z be arbitrary words in Σ^*. Consider the following paths in $\Gamma(\mathcal{P})$:

$$(u, \ell_1 \to r_1, z\ell_2 v)(ur_1 z, \ell_2 \to r_2, v), \tag{2}$$

$$(u\ell_1 z, \ell_2 \to r_2, v)(u, \ell_1 \to r_1, zr_2 v). \tag{3}$$

It is easy to see that $(u\ell_1 z\ell_2 v, ur_1 zr_2 v)$-diagrams corresponding to these paths are equal. This diagram is shown on the following picture:



This situation hints to a homotopy relation on the set of paths in the graph $\Gamma(\mathcal{P})$: paths (2) and (3) should be called homotopic. In order to define the homotopy relation we need the structure of a 2-complex on $\Gamma(\mathcal{P})$. For every 5-tuple $(u, \ell_1 = r_1, z, \ell_2 = r_2, v)$, where $u, v, z \in \Sigma^*$, $(\ell_1 = r_1), (\ell_2 = r_2) \in \mathcal{R}$ we have a 2-cell whose defining path is $p_1 p_2^{-1}$ where p_1, p_2 are the paths (2) and (3), respectively. The resulting 2-complex is called the *Squier complex* of the semigroup presentation $\mathcal{P}$. It is denoted by $\mathcal{K}(\mathcal{P})$. It was implicitely defined by Squier in [31]. The same complex was independently constructed by Kilibarda [17, 18] and Pride [26]. The important role of this complex is justified by the fact that equal diagrams over $\mathcal{P}$ correspond to homotopic paths in $\mathcal{K}(\mathcal{P})$. The following Kilibarda's theorem [17, 18] plays an important role in this paper: *The diagram group $\mathcal{D}(\mathcal{P}, w)$ is isomorphic to the fundamental group $\pi_1(\mathcal{K}, w)$ of the Squier complex $\mathcal{K} = \mathcal{K}(\mathcal{P})$.*

2 Diagram Product of Groups

In [12], we considered several group-theoretical operations such that the class of diagram groups is closed under them. These operations were: finite direct products (result due to Kilibarda [17]), any free products, and also some special operation $\bullet$ which we used for constructing an example of a diagram group that was finitely generated but not finitely presented. In this Section we introduce a quite general operation on groups, the *diagram product*. We show that the class of diagram groups is closed under this operation. All the above listed constructions are partial cases of this new operation. We also consider some concrete applications of this construction. They will be essentially used in the later Sections.

Let us recall the definition of a graph. A *graph* (in the sense of Serre [30]) is an ordered tuple $\Gamma = \langle\, V, E, ^{-1}, \iota, \tau\,\rangle$ where V, E are disjoint sets, $^{-1}$ is an involution on E, ι, τ are mappings from E to V. The following axioms hold:

- $e^{-1} \neq e$ for any $e \in E$;
- $\iota(e^{-1}) = \tau(e)$, $\tau(e^{-1}) = \iota(e)$.

Elements of the sets V and E are called *vertices* and *edges* of the graph respectively. If $e \in E$, then $\iota(e)$ is called the *initial vertex* of the edge e, and $\tau(e)$ is called the *terminal vertex* of the edge e.

A *path* on the graph Γ is either a vertex, or a nonempty sequence of edges e_1, e_2, $\dots$, e_n such that $\tau(e_i) = \iota(e_{i+1})$ for each $i = 1, \dots, n-1$. Usually a path is written in the form $p = e_1 e_2 \dots e_n$. If a path p consists of a vertex v, then it is called an *empty path* and we denote it by 1_v. If $p = e_1 e_2 \dots e_n$ is a path, then the *inverse path* p^{-1} is the path $e_n^{-1} e_{n-1}^{-1} \dots e_1^{-1}$. An empty path coincides with its inverse. A path p is called *closed* whenever $\iota(p) = \tau(p)$.

An *orientation* on the graph Γ is a subset E^+ of the set E of all edges such that, for any edge $e \in E$, there is exactly one of the edges e, e^{-1} that belongs to E^+. The edges in E^+ are called *positive* and the edges in $E^- = E \setminus E^+$ are called *negative*. A path on an oriented graph is called *positive* whenever it involves positive edges only. (An empty path is always positive.) For any path p, there are defined its *initial vertex* $\iota(p)$ and its *terminal vertex* $\tau(p)$: if $p = 1_v$, then $\iota(p) = \tau(p) = v$; if $p = e_1 \dots e_n$, then $\iota(p) = \iota(e_1)$, $\tau(p) = \tau(e_n)$. For any two paths p, q such that $\tau(p) = \iota(q)$, one can naturally define a *product* $p \cdot q$ of the paths p and q: for $p = e_1 \dots e_n$, $q = f_1 \dots f_m$ we put $p \cdot q = e_1 \dots e_n f_1 \dots f_m$. If p (q) is empty, then $p \cdot q = q$ ($p \cdot q = p$).

An *orineted graph* is by definition a graph Γ with a fixed orientation E^+. It is clear that any graph admits an orientation.

The concept of a graph of groups will play an important role. Let us have an oriented graph Γ, where E^+ is the set of positive edges. We say that a *graph of groups* structure on the graph Γ is given whenever to each edge $e \in E^+$ we assign a group G_e, to each vertex $v \in V$ we assign a group G_v, and we fix embeddings $\iota_e\colon G_e \to G_{\iota(e)}$, $\tau_e\colon G_e \to G_{\tau(e)}$ for any $e \in E^+$.

In the construction described below, we will have a 2-complex structure on Γ together with the graphs of groups structure. This means that we have a set F which is disjoint from V and E. This set is called the *set of 2-cells*. We also have a mapping that assigns a closed path in Γ to each element in F. This path is called the *defining path* of the 2-cell. Given a 2-complex, we define the homotopy relation on the set of paths in a standard way. Also one can define the concept of the fundamental group of $\mathcal{K}$ with basepoint w. We denote this group by $\pi_1(\mathcal{K}, w)$.

We will consider 2-complexes that have a graph of groups structure on their 1-skeletons. We shall call such structures *2-complexes of groups*. The concept of a 2-complex of groups already exists and it is used widely in many papers (see [14]). Every 2-complex of groups in our sense is a 2-complex of groups in the sense of [14], but not vice versa. (In general, a 2-complex of groups is a structure that has not only vertex groups G_v ($v \in V$) and edge groups G_e ($e \in E$) but also cell groups of the form G_f ($f \in F$) that are assigned to 2-cells. In our situation all the cell groups G_f are trivial.)

So, let $\mathcal{G}$ be a 2-complex of groups. Now we define the fundamental group of $\mathcal{G}$. One can define it in different ways. We will define it as a fundamental group of an ordinary 2-complex ${}_{\mathcal{K}}(\mathcal{G})$ with a basepoint. Here is the description of the complex ${}_{\mathcal{K}}(\mathcal{G})$.

We add new edges and new 2-cells to the 2-complex $\mathcal{K}$. For any vertex $v \in V$ and for any element $g \in G_v$ we add an edge denoted by g_v that has v as both initial and terminal vertex. The new 2-cells are of two types. 2-cells of the first type have defining paths $g_v h_v (gh)_v^{-1}$ for any vertex $v \in V$ and for any elements $g, h \in G_v$. 2-cells of the second type have defining paths $e^{-1} g_{\iota(e)} e h_{\tau(e)}^{-1}$, where $g = \iota_e(x)$, $h = \tau_e(x)$, $x \in G_e$, $e \in E^+$. (The 2-cells of the second type correspond to all pairs of the form (e, x), where $e \in E^+$, $x \in G_e$.) Recall that ι_e, τ_e are embeddings of the group G_e into the groups $G_\iota(e)$, $G_{\tau(e)}$, respectively. The 2-complex obtained from $\mathcal{K}$ by adding new 2-cells will be denoted by ${}_{\mathcal{K}}(\mathcal{G})$. For any vertex $v \in V$, the fundamental group $\pi_1({}_{\mathcal{K}}(\mathcal{G}), v)$ will be called the *fundamental group* of *2-complex of groups* $\mathcal{G}$ with *basepoint* v. It will be denoted by $\pi_1(\mathcal{G}, v)$.

A standard way to compute the fundamental group of a 2-complex (see [33]) can be also applied to compute the fundamental group of a 2-complex of groups. Let we have a structure of a graph of groups $\mathcal{G}$ on the 1-skeleton of a 2-complex $\mathcal{K}$. Consider the connected component $\mathcal{K}_w$ of $\mathcal{K}$ that contains vertex w and let us choose some maximal subtree $\mathcal{T}_w$ in this component. It will also be a maximal subtree of the connected component of the new 2-complex $_{\mathcal{K}}(\mathcal{G})$ that contains vertex w. Let us take the set of positive edges E_w^+ of this connected component together with elements of the form g_v, $v \in V_w$, $g \in G_v$, where V_w is the set of vertices of this connected component. The union of these sets is the set of generators of the fundamental group. For defining relations, we take all relations of the form $e = 1$, where e belongs to the tree $\mathcal{T}$, and also all relations of the form $r = 1$, where r is the defining path of any 2-cell of the 2-complex $_{\mathcal{K}}(\mathcal{G})$. The group given by the described presentation is isomorphic to the fundamental group $\pi_1(\mathcal{G}, w)$ of our 2-complex of groups.

Let us give one more equivalent description. It is clear that the fundamental group $\pi_1(\mathcal{K}, w)$ with basepoint w of the original 2-complex $\mathcal{K}$ can be computed in the same way, choosing $\mathcal{T}_w$ as a maximal subtree of the connected component of w. Now each edge e of this component uniquely defines an element in $\pi_1(\mathcal{K}, w)$, namely, the equivalence class of the path $p_{\iota(e)} e p_{\tau(e)}^{-1}$, where p_v denotes the geodesic path from w to v in the subtree $\mathcal{T}_w$. Then

$$\pi_1(\mathcal{G}, w) \cong \mathop{*}_{v} G_v * \pi_1(\mathcal{K}, w) \Big/ \mathcal{N}, \tag{4}$$

where the free product of groups G_v is taken over all vertices v of the connected component of $\mathcal{K}$ that contains w, and $\mathcal{N}$ is the normal closure of the following relations:

$$g_{\iota(e)}^{e} = h_{\tau(e)} \text{ for every } e \in E_w^+,\ x \in G_e,\ \text{ where } g = \iota_e(x),\ h = \tau_e(x). \tag{5}$$

This description will be often used below.

Let us give one of the main definitions.

Definition 3 Let X be an alphabet, let H_x $(x \in X)$ be an arbitrary family of groups and let $\mathcal{Q} = \langle X \mid \mathcal{S} \rangle$ be a semigroup presentation, $w \in X^+$. Consider the Squier complex $\mathcal{K} = \mathcal{K}(\mathcal{Q})$ and introduce the structure of graph of groups on its 1-skeleton in the following way. Let E^+ be the set of all positive edges of $\mathcal{K}$, that is the set of triples of the form $e = (u, s \to t, v)$, where $u, v \in X^*$, $(s = t) \in \mathcal{S}$. For any word $z = x_1 x_2 \dots x_n$, where $x_1, x_2, \dots, x_n \in X$, let

$$H_u = H_{x_1} \times H_{x_2} \times \cdots H_{x_n};$$

if u is empty, then $H_u = 1$. For any vertex $u \in \mathcal{K}$ let $G_u = H_u$; for any edge $e = (u, s \to t, v) \in E^+$, where $u, v \in X^*$, $(s = t) \in \mathcal{S}$, let $G_e = H_u \times H_v$. We have the natural embeddings $\iota_e \colon G_e \to G_{\iota(e)}$ as the embedding $H_u \times H_v \to H_{usv} = H_u \times H_s \times H_t$ and $\tau_e \colon G_e \to G_{\tau(e)}$ as an embedding $H_u \times H_v \to H_{utv} = H_u \times H_t \times H_t$. This gives us a 2-complex of groups, which will be denoted by $\mathcal{K}_H$. The fundamental group $\pi_1(\mathcal{K}_H, w)$ of this 2-complex of groups with basepoint w is called the *diagram product* of the family $H_X = \{ H_x\ (x \in X) \}$ of groups over the presentation $\mathcal{Q} = \langle X \mid \mathcal{S} \rangle$ with base w. It will be denoted by $\mathcal{D}(H_X; \mathcal{S}, w)$.

It is easy to see that the diagram product $\mathcal{D}(H_X;\mathcal{S},w)$ coincides with the diagram group $\mathcal{D}(\mathcal{Q},w)$ in the case when the groups H_x are trivial for all $x\in X$.

The main result about this construction is that the diagram product of any family of diagram groups over a semigroup presentation is again a diagram group. Let us formulate this result in its general form giving the description of the presentation, for which the corresponding diagram group is the diagram product.

Theorem 4 *Let $\mathcal{Q}=\langle\, X \mid \mathcal{S}\,\rangle$ be a semigroup presentation, $w\in X^+$. To each $x\in X$ we assign a diagram group $G_x=\mathcal{D}(\mathcal{P}_x,w_x)$, where $\mathcal{P}_x=\langle\,\Sigma_x \mid \mathcal{R}_x\,\rangle$ are semigroup presentations, $w_x\in\Sigma_x^+$ $(x\in X)$. Let $A=\{\,a_x \mid x\in X\,\}$ be some alphabet. Assume that the alphabets X, A, Σ_x $(x\in X)$ are disjoint. Let*

$$\Sigma=\bigcup_{x\in X}\Sigma_x,\quad \mathcal{R}=\bigcup_{x\in X}\mathcal{R}_x,\quad \mathcal{W}=\bigcup_{x\in X}\{\,x=a_xw_xa_x\,\}.$$

Consider the presentation

$$\mathcal{P}=\langle\, X\cup\Sigma\cup A \mid \mathcal{S}\cup\mathcal{R}\cup\mathcal{W}\,\rangle.$$

We claim that the diagram group $\mathcal{D}(\mathcal{P},w)$ is isomorphic to the diagram product $\mathcal{D}(G_X;\mathcal{S},w)$ of the family $G_X=\{\,G_x\ (x\in X)\,\}$ of groups over the presentation $\mathcal{Q}=\langle\, X \mid \mathcal{S}\,\rangle$ with base w.

In particular, the diagram product of any family of groups over a semigroup presentation is a diagram group.

Let us consider a geometric description of this construction. In the above notation, each group G_x is isomorphic to the diagram group over the presentation $\hat{\mathcal{P}}_x$ that is obtained from $\mathcal{P}_x$ by adding letters x, a_x to the alphabet Σ_x and adding new relation $x=a_xw_xa_x$ to the set $\mathcal{R}_x$. We have $G_x\cong\mathcal{D}(\hat{\mathcal{P}}_x,x)$. Let us take any (w,w)-diagram over $\mathcal{Q}$ and consider some of its edges. Let $x\in X$ be its label. We can cut the diagram along this edge and insert any (x,x)-diagram over $\hat{\mathcal{P}}_x$ in the resulting hole. We can do this with all edges of the diagram. (If we insert trivial (x,x)-diagram, then nothing changes.) After these transformations, we obtain some (w,w)-diagram over $\mathcal{P}$. One can show that diagrams obtained in this way form the whole group $\mathcal{D}(\mathcal{P},w)$. From this point of view, diagrams that represent elements in $\mathcal{D}(\mathcal{P},w)$ are obtained from (w,w)-diagrams over $\mathcal{Q}$ by insertions of (x,x)-diagrams, which represent elements in G_x.

Proof. Let us construct the Squier complex for the presentation $\mathcal{P}$. By Kilibarda's Theorem, the fundamental group of this complex with basepoint w is isomorphic to the diagram group $\mathcal{D}(\mathcal{P},w)$. Our goal is to transform the Squier complex into some new 2-complex with the same fundamental group. We will need to check that it will be isomorphic to the fundamental group of a certain 2-complex of groups, that is, to the diagram products of our groups.

Let $\mathcal{K}_w(\mathcal{P})$ be the connected component of the Squier complex over $\mathcal{P}$, which contains the vertex w. A vertex v in the same component is an arbitrary word that equals w modulo

$\mathcal{P}$. This word can be uniquely decomposed into the product $v = v_1 \ldots v_\mu$ ($\mu = \mu(v)$) of subwords $v_1, \ldots, v_\mu$ in such a way that each of them will be either a letter in X, or a word of the form $a_x u a_x$, where u is a word over Σ_x that equals w_x modulo $\mathcal{P}_x$. The words $v_1, \ldots, v_\mu$ will be called the *factors* of the word v. To each factor v_i ($1 \le i \le \mu$), we assign a letter in the alphabet X. This letter will be denoted by $\pi(v_i)$. If $v_i \in X$, then we put $\pi(v_i) = v_i$ and if v_i has a form $a_x u a_x$, where u equals w_x modulo $\mathcal{P}_x$, then we put $\pi(v_i) = x$. Let us extend the function π, putting by definition $\pi(v) = \pi(v_1) \ldots \pi(v_\mu)$ for any word v that equals w modulo $\mathcal{P}$. The function π will be called the *projection*.

There is a natural two-sided action of the free monoid $M = (X \cup \Sigma \cup A)^*$ on the Squier complex. It can be defined by the following rule: for any $m_1, m_2 \in M$ and for any vertex v of the Squier complex, let $m_1 * v * m_2$ be the vertex $m_1 v m_2$ and let $m_1 * e * m_2$ be the edge $(m_1 u, p \to q, v m_2)$, for any edge $e = (u, p \to q, v)$. The images of a given subgraph of $\mathcal{K}(\mathcal{P})$ under this action will be called the *shifts* of this subgraph.

Let $\hat{\mathcal{R}}_x = \mathcal{R}_x \cup \{x = a_x w_x a_x\}$. We introduce a presentation $\hat{\mathcal{P}}_x = \langle\, \Sigma_x, x, a_x \mid \hat{\mathcal{R}}_x \,\rangle$ for all $x \in X$. It is clear that $\mathcal{D}(\hat{\mathcal{P}}_x, x) \cong G_x$. Let v be a vertex of the complex $\mathcal{K}_w(\mathcal{P})$ and let $v = v_1 \ldots v_\mu$ be the decomposition of the word v into factors. It is obvious that for any $1 \le i \le \mu$, the letter $\pi(v_i) = x$ is equal to the word v_i modulo $\hat{\mathcal{P}}_x$. Also it is clear that v equals $\pi(v)$ modulo $\mathcal{P}$. Note that Squier complexes for presentations $\hat{\mathcal{P}}_x$ and their shifts can be regarded as subcomplexes of the Squier complex of $\mathcal{P}$.

For each $x \in X$ we choose a maximal subtree $\mathcal{T}_x$ in the connected component $\mathcal{K}(\hat{\mathcal{P}}_x, x)$ of the Squier complex of the presentation $\hat{\mathcal{P}}_x$ that contains the vertex x. Let us also choose a maximal subtree $\mathcal{T}_Q$ in the connected component $\mathcal{K}(\mathcal{Q}, w)$ of the Squier complex of the presentation $\mathcal{Q}$ that contains vertex w. Let v be an arbitrary vertex of the complex $\mathcal{K}_w(\mathcal{P})$ and let $v = v_1 \ldots v_\mu$ be the decomposition of v into factors. Let $x_i = \pi(v_i)$ ($1 \le i \le \mu$). Consider the following subgraphs of $\mathcal{K}_w(\mathcal{P})$:

$$1 * \mathcal{T}_{x_1} * x_2 \ldots x_\mu,\ \ v_1 * \mathcal{T}_{x_2} * x_3 \ldots x_\mu, \ldots,\ \ v_1 \ldots v_{\mu-1} * \mathcal{T}_{x_\mu} * 1. \qquad (6)$$

Now consider the subgraph $\mathcal{T}$ of $\mathcal{K}_w(\mathcal{P})$ that is a union of subgraphs (6) for all vertices v that are equal to w modulo $\mathcal{P}$, together with the subgraph $\mathcal{T}_Q$. Let us prove that $\mathcal{T}$ is a maximal subtree of $\mathcal{K}_w(\mathcal{P})$.

First of all we will establish that $\mathcal{T}$ is a connected subgraph that contains all vertices of $\mathcal{K}_w(\mathcal{P})$, that is, for any vertex v of our component, we will find a path in $\mathcal{T}$ from w to v. Let $v = v_1 \ldots v_\mu$ be the decomposition of v into a product of factors. By p we denote the geodesic path in $\mathcal{T}_Q$ from w to $\pi(v) = x_1 \ldots x_\mu$, where $x_i = \pi(v_i)$ for all i from 1 to μ. For each i, let p_i be the geodesic path from x_i to v_i in the graph $\mathcal{T}_{x_i}$. For each i from 1 to μ we consider the path $\tilde{p}_i = v_1 \ldots v_{i-1} * p_i * x_{i+1} \ldots x_\mu$. Obviously, it connects vertices $v_1 \ldots v_{i-1} x_i \ldots x_\mu$ and $v_1 \ldots v_i x_{i+1} \ldots x_\mu$ in the graph $\mathcal{T}$. The product $p\tilde{p}_1 \ldots \tilde{p}_\mu$ is a path in $\mathcal{T}$ from w to $v_1 \ldots v_\mu = v$.

Now let us prove that $\mathcal{T}$ has no nontrivial cycles. We argue by contradiction. Suppose that a nontrivial cycle exists. If it does not consist of edges that belong to subgraphs of the form (6), then it has an edge from $\mathcal{T}_Q$. Since $\mathcal{T}_Q$ has no nontrivial cycles, our cycle must contain edges from subgraphs of the form (6). Hence our cycle has a nontrivial cyclic subpath ρ that is a loop at some vertex in $\mathcal{K}_w(\mathcal{Q})$ and all its edges are from subgraphs of the form (6). Let us make a simple but important observation: the endpoints of each edge that belong to any shift of the subcomplex $\mathcal{K}(\hat{\mathcal{P}}_x)$ $(x \in X)$, have equal projection. Indeed, applying relations of the form $x = a_x w_x a_x$ does not change the projection, and applying relations from $\mathcal{R}_x$ occurs within a factor of the form $a_x u a_x$, where u is a word over Σ_x. This also does not change the projection, (in the last case one can see the role of the auxiliary alphabet A). Thus projections of all vertices of the cyclic path ρ coincide, that is, ρ is a nontrivial cycle in $\mathcal{T}$ that consists of edges from subgraphs of the form (6). So in any case there is a nontrivial cycle ρ with the above property. Without loss of generality, one can assume that ρ does not contain occurrences of adjacent edges that are mutually inverse.

Let $v = v_1 \dots v_\mu$ be the decomposition of v into factors, where ρ is the loop at v. Each edge of the path ρ touches exactly one of the factors, as we could see above. Let j be the greatest number such that an edge of ρ touches jth factor. Let $x_i = \pi(v_i)$ $(1 \le i \le \mu)$. It follows from the structure of subgraphs (6) that $v_i = x_i \in X$ for all $j < i \le \mu$. Since the jth factor occurs in the process of application of relations from $\hat{\mathcal{R}}_j$ to it, the path ρ or one of its cyclic shifts has a maximal subpath ρ' that consists of edges that touch the jth factor only. Let v' and v'' be the initial and the terminal points of ρ' respectively. We claim that $v' = v''$. Suppose this is not true. It is clear that v' and v'' differ by the jth factor only and so one can say that the jth factor is not equal to x_j either in v' or in v''. Assume that the jth factor of v'' is not equal x_j. By our assumption, ρ' has fewer edges than ρ (otherwise $v' = v''$ automatically). So there is an edge e such that $\rho' e$ is a subpath of some cyclic shift of the path ρ. Since ρ' was chosen maximal, the edge e does not touch the jth factor. It also cannot touch a factor with a number greater than j because j is maximal with this property. But it also cannot touch a factor with a number less than j because it belongs to a subgraph of the form (6), and the jth factor of the initial point of e does not belong to X, a contradiction. So ρ' is a nontrivial cycle that belongs to a shift of the tree $\mathcal{T}_{x_j}$. However, this is impossible since a shift of a tree is a tree itself. This contradiction shows that $\mathcal{T}$ has no nontrivial cycles. Applying what we have said above, we conclude that $\mathcal{T}$ is a maximal subtree in $\mathcal{K}_w(\mathcal{P})$.

Now we need to calculate the fundamental group $G = \pi_1(\mathcal{K}_w(\mathcal{P}))$ by using the maximal subtree $\mathcal{T}$. All edges of the complex $\mathcal{K}_w(\mathcal{P})$ are regarded as elements of the group G, and the edges from $\mathcal{T}$ equal the identity in G. Paths in this complex, regarded as products of edges, are just elements of the group G. To understand how the other relations in G look like, we need to describe the 2-cells in $\mathcal{K}_w(\mathcal{P})$. First of all let us mention some important property. Recall that M is the free monoid over the alphabet of the presentation $\mathcal{P}$, and M acts both from the left and from the right on the complex $\mathcal{K}(\mathcal{P})$. Let s, t, u, v be elements in M, each decomposed into the product of factors, and let s equals t modulo $\mathcal{P}$, usv equals w modulo $\mathcal{P}$. Let us take an arbitrary path p in $\mathcal{K}(\mathcal{P})$ that connects vertices s and t. It is clear that the paths $u*p*v$ and $\pi(u)*p*\pi(v)$ belong to $\mathcal{K}_w(\mathcal{P})$. We claim that the following equality holds

$$u * p * v = \pi(u) * p * \pi(v) \tag{7}$$

in the group G. This is what we are going to prove. To prove that, one can consider the contours of the corresponding 2-cells as words in the generators of the group G which are equal to the identity in G. However, we think that one can check equality (7) easier, using the Kilibarda Theorem. Namely, to prove the equality (7), it suffices to use the fact that G is isomorphic to the diagram group over $\mathcal{P}$ with base $\pi(usv)$. So let us find the diagrams over $\mathcal{P}$ that represent the elements in G from both sides of equality (7), and then let us check that the diagrams are equal.

Let $x = x_1 \dots x_m$ be any word in M decomposed into the product of its factors. For any i from 1 to m let q_i be the geodesic path from $\pi(x_i)$ to x_i in the tree $\mathcal{T}_{\pi(x_i)}$. Then

$$p_x = (1 * q_1 * \pi(x_2 \dots x_m))(x_1 * q_2 * \pi(x_3 \dots x_m)) \dots (x_1 \dots x_{m-1} * q_m * 1)$$

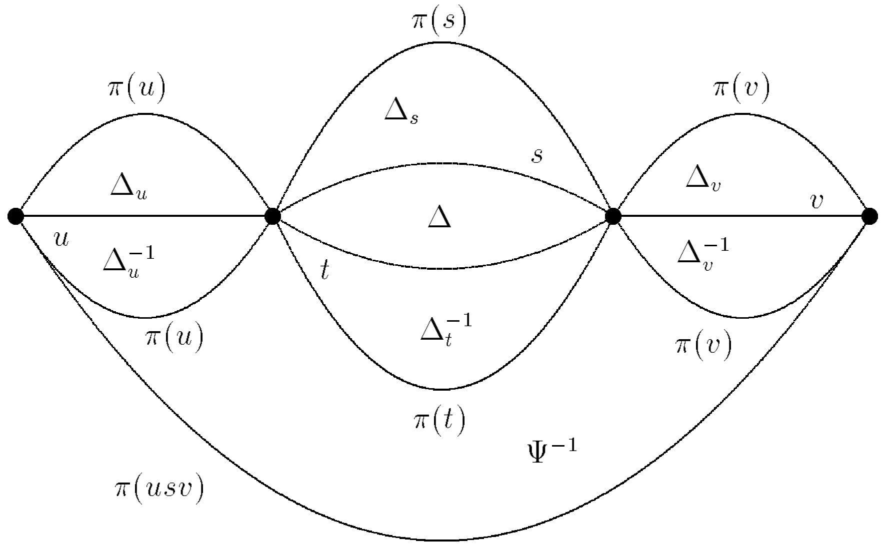
is a path from $\pi(x)$ to x. Let Δ_x be the diagram represented by it. Let us consider such paths and diagrams for all $x \in \{s, t, u, v\}$. Also let q be the geodesic path from $\pi(usv)$ to $\pi(utv)$ in the tree $\mathcal{T}_Q$, and let Δ, Ψ be the diagrams represented by p, q, respectively. To find the spherical diagram with base $\pi(usv)$ represented by the path $u * p * v$ (via the isomorphism of the diagram group and the group G), one needs to concatenate three diagrams: the Δ_1 that corresponds to the path in the tree $\mathcal{T}$ from $\pi(usv)$ to usv, the diagram $\Delta_2 = \varepsilon(u) + \Delta + \varepsilon(v)$ (that corresponds to the path $u * p * v$), and the diagram Δ_3 that corresponds to the path in the tree $\mathcal{T}$ from utv to $\pi(usv)$. So consider the path $(1 * p_u * \pi(sv))(u * p_s \pi(v))(us * p_v * 1)$. It follows from the description of subgraphs (6) that this path is contained in $\mathcal{T}$. It corresponds to the diagram

$$\Delta_1 = (\Delta_u + \varepsilon(\pi(sv)))(\varepsilon(u) + \Delta_s + \varepsilon(\pi(v)))(\varepsilon(us) + \Delta_v).$$

Further, the path $(ut * p_v^{-1} * 1)(u * p_t^{-1} * \pi(v))(1 * p_u^{-1} * \pi(tv))$ is contained in $\mathcal{T}$. Multiplying it by the path q^{-1} on the right, we obtain the path in $\mathcal{T}$ from utv to $\pi(usv)$. This path is represented by the diagram

$$\Delta_3 = (\varepsilon(ut) + \Delta_v^{-1})(\varepsilon(u) + \Delta_t^{-1} + \varepsilon(\pi(v)))(\Delta_u^{-1} + \varepsilon(\pi(tv)))\Psi^{-1}.$$

Let us now multiply the diagrams Δ_1, Δ_2 and Δ_3. It is easy to see that the subdiagram Δ_u cancels with Δ_u^{-1} in this product, and Δ_v cancels with Δ_v^{-1} (see the picture below).



After cancelling Δ_u and Δ_u^{-1}, Δ_v and Δ_v^{-1}, we obtain a diagram that is a product

$$(\varepsilon(\pi(u)) + \Delta_s + \varepsilon(\pi(v)))(\varepsilon(\pi(u)) + \Delta + \varepsilon(\pi(v)))(\varepsilon(\pi(u)) + \Delta_t^{-1} + \varepsilon(\pi(v)))\Psi^{-1}. \qquad (8)$$

Repeating the arguments of the above paragraph for the path $\pi(u) * p * \pi(v)$, it is easy to see that this path is represented by the diagram (8) in the diagram group over $\mathcal{P}$ with the base $\pi(usv)$. This proves the equality (7).

Let us consider an arbitrary edge $(u, s \to t, v)$ of the complex $\mathcal{K}_w(\mathcal{P})$. Let $(s = t) \in \mathcal{S}$. The words u, v can be decomposed into products of factors, and the equality $(u, s \to t, v) = (\pi(u), s \to t, \pi(v))$ holds in G. The right-hand side of this equality can be regarded as an element of the group $\pi_1(\mathcal{K}_w(\mathcal{Q}))$, where $\mathcal{T}_Q$ is the maximal subtree in $\mathcal{K}_w(\mathcal{Q})$. Now let $(s = t) \notin \mathcal{S}$. In this case there exists an element $x \in X$ and words u_1, v_1, u_2, v_2 such that $u = u_1u_2$, $v = v_2v_1$, where u_1 (v_1) is the maximal prefix (suffix) of the word u (resp. v) that can be decomposed into a product of factors. Here u_2sv_2 equals x modulo $\mathcal{P}$. Then by (7) we have the equality $e = (u_1u_2, s \to t, v_2v_1) = \pi(u_1) * (u_2, s \to t, v_2) * \pi(v_1)$. For any $x \in X$, let us consider the fundamental group $\pi_1(\mathcal{K}(\hat{\mathcal{P}}_x), x) \cong G_x$ that can be calculated using the maximal subtree $\mathcal{T}_x$ in the connected component of the Squier complex over $\mathcal{P}_x$ that contains x. The edges of this component will be thus the elements of a group isomorphic to G_x so the edge e will belong to an isomorphic copy of this group that is generated by edges obtained as a result of shifts. Namely, let $U, V \in X^*$, $x \in X$. We consider the group denoted by $U * G_x * V$. It is generated by edges of the form $U * f * V$, where f runs over edges that generate the group $\pi_1(\mathcal{K}(\hat{\mathcal{P}}_x), x) \cong G_x$. In this sense, the edge e belongs to the group $\pi(u_1) * G_x * \pi(v_1)$. The argument of this paragraph can be summarized as follows: the group G is generated by the subgroup $\pi_1(\mathcal{K}(\mathcal{Q}), w) \cong \mathcal{D}(\mathcal{Q}, w)$ and groups of the form $u * G_x * v$, where $u, v \in X^*$, $x \in X$, and uxv equals w modulo $\mathcal{P}$.

Now it remains to analyze all 2-cells of $\mathcal{K}_w$ and to find out what will be the relations between the generators of G described above. According to the description of 2-cells in a Squier complex given in Section 1, let a 2-cell be given by a 5-tuple $(u, \ell_1 \to r_1, z, \ell_2 \to r_2, v)$, where (ℓ_1, r_1), (ℓ_2, r_2) belong to $\mathcal{R} \cup \mathcal{S} \cup \mathcal{W}$. Note that the word $u\ell_1z\ell_2v$ equals w modulo $\mathcal{P}$. Let us consider several cases depending on the defining relations involved. The relation between edges that is obtained from the given 2-cell, has the form

$$(u, \ell_1 \to r_1, z\ell_2v)(ur_1z, \ell_2 \to r_2, v) = (u\ell_1z, \ell_2 \to r_2, v)(u, \ell_1 \to r_1, zr_2v). \qquad (9)$$

a) Let (ℓ_1, r_1), (ℓ_2, r_2) both belong to $\mathcal{S}$. Then each of the words u, v, z can be decomposed into the product of factors. Using equality (7), one can replace in (9) the words u, v, z by their projections (we use the fact that each of the words ℓ_j, r_j $(j = 1, 2)$ coincides with its projection). Thus one can assume that the words u, v, z in (9) belong to X^*. Then (9) is a defining relation of the group $\pi_1(\mathcal{K}(\mathcal{Q}), w) \cong \mathcal{D}(\mathcal{Q}, w)$ (calculated by using the maximal subtree $\mathcal{T}_Q$).

b) Suppose that none of the relations (ℓ_1, r_1), (ℓ_2, r_2) belongs to $\mathcal{S}$. Suppose also that these relations are applied to different factors of the word $u\ell_1z\ell_2v$. This means that there exist letters $x, y \in X$ and decompositions of the form $u = u_1u_2$, $z = z'z_0z''$, $v = v_2v_1$, where u_1 is the maximal prefix of u that is a product of factors, v_1 is the maximal suffix of v that is a product of factors, and z_0 is a maximal subword of the word z that is a product of factors. (It is not hard to see that this word can be found uniquely.) Here x equals $u_2\ell_1 z'$ and y equals $z''\ell_2 v_2$ (equalities are considered modulo $\hat{\mathcal{P}}_x$ and $\hat{\mathcal{P}}_y$, respectively). Let us substitute the decompositions of the words u, v, z in the equality (9) using the fact that $\pi(u_2\ell_1 z') = \pi(u_2 r_1 z') = x$, $\pi(z''\ell_2 v_2) = \pi(z'' r_2 v_2) = y$. We obtain that the elements $\pi(u_1) * (u_2, \ell_1 \to r_1, z') * \pi(z_0) y \pi(v_1)$ $\pi(u_1) x \pi(z_0) * (z'', \ell_2 \to r_2, v_2) * \pi(v_1)$ commute. The first of them belongs to the group $\pi(u_1) * G_x * \pi(z_0) y \pi(v_1)$, and the second one belongs to the group $\pi(u_1) x \pi(z_0) * G_y * \pi(v_1)$. Let U, V, Z be arbitrary words over X and let $x, y \in X$ be arbitrary letters such that the word $UxZyV$ equals w modulo $\mathcal{P}$. We can conclude that any element in $U * G_x * ZyV$ commutes with any element in $UxZ * G_y * V$ since for any edges e and f that belong to the generating sets of the groups G_x and G_y respectively, one can find a suitable 2-cell of the form described above in such a way that the defining relations obtained from it will be the relation of commutativity of $U * e * ZyV$ and $UxZ * f * V$. Thus we get relations of the form $[U * G_x * ZyV, UxZ * G_y * V] = 1$, where $UxZyV$ equals w modulo $\mathcal{P}$, $U, V, Z \in X^*$, $x, y \in X$.

c) Again, let none of the relations (ℓ_1, r_1), (ℓ_2, r_2) belong to $\mathcal{S}$ but assume now that the relations $(\ell_1 = r_1)$ and $(\ell_2 = r_2)$ are applied to the same factor of the word $u\ell_1 z\ell_2 v$. This means that there exists a letter $x \in X$ and decompositions $u = u_1 u_2$, $v = v_2 v_1$, where u_1 is the maximal prefix of the word u that is a product of factors, v_1 is the maximal suffix of the word v that is a product of factors. Now x equals $u_2\ell_1 z\ell_2 v_2$ modulo $\hat{\mathcal{P}}_x$. Consider a 2-cell of the Squier complex over $\hat{\mathcal{P}}_x$ that corresponds to the 5-tuple $(u_2, \ell_1 \to r_1, z, \ell_2 \to r_2, v_2)$. All cells of this form lead to the defining relations of a group isomorphic to G_x. Acting on this 2-cell by the element $\pi(u_1)$ on the left and by the element $\pi(v_1)$ on the right, we get a 2-cell of the complex $\mathcal{K}_w(\mathcal{P})$. The relation written on its contour is equivalent to (9) if one takes the equality (7) into account. Thus we get the defining relations of all groups of the form $U * G_x * V$, where UxV equals w modulo $\mathcal{P}$, $U, V \in X^*$, $x \in X$.

d) Suppose that one of the relations (ℓ_1, r_1), (ℓ_2, r_2) belongs to $\mathcal{S}$ and the other one does not. First of all, let $(\ell_1, r_1) \in \mathcal{S}$. Then we have decompositions of the form $z = z_0 z''$, $v = v_2 v_1$, where z_0, v_1 are products of factors that are chosen to be minimal with respect to this property, as above. Now $z''\ell_2 v_2$ equals x modulo $\hat{\mathcal{P}}_x$ for some letter $x \in X$. Let $f = (z'', \ell_2 \to r_2, v_2)$. Substituting the decompositions of words v, z in (9), taking into account that $\pi(z''\ell_2 v_2) = \pi(z'' r_2 v_2) = x$ and applying (7), we obtain the following equality:

$$(\pi(u), \ell_1 \to r_1, \pi(z_0) x \pi(v_1)) \cdot (\pi(u) r_1 \pi(z_0) * f * \pi(v_1)) =$$
$$(\pi(u)\ell_1 \pi(z_0) * f * \pi(v_1)) \cdot (\pi(u), \ell_1 \to r_1, \pi(z_0) x \pi(v_1)) \quad .$$

Thus for each $x \in X$ and for any words U, Z, V over X such that $(\ell_1, r_1) \in \mathcal{S}$ and $U\ell_1 ZxV$ equals w modulo $\mathcal{P}$, we obtain the relations

$$U r_1 Z * f * V = (U\ell_1 Z * f * V)^e, \tag{10}$$

where $e = (U, \ell_1 \to r_1, ZxV)$, and f runs over the generating set of the group $\pi_1(\hat{\mathcal{P}}_x, x) \cong G_x$.

Analogously, if $(\ell_2, r_2) \in \mathcal{S}$, then we get the relations

$$U * f * Zr_2V = (U * f * Z\ell_2V)^e, \tag{11}$$

where U, Z, V are words over X, $x \in X$, $e = (UxZ, \ell_2 \to r_2, V)$, and f runs over the generating set of the group $\pi_1(\hat{\mathcal{P}}_x, x) \cong G_x$.

To compute the diagram product, let us define a structure of a graph of groups on the 1-skeleton of the Squier complex $\mathcal{K}(\mathcal{Q})$. Our diagram product is the fundamental group of the corresponding 2-complex of groups. We will apply the above described procedure of computing a fundamental group of a 2-complex of groups and we will then compare it with the presentation of G. Let U, V be words over X, $x \in X$. By $H(U \cdot x \cdot V)$ we denote the group $U * G_x * V$ isomorphic to G_x. Then, for any word $u = u_1 \dots u_m$, where $u_i \in X$ $(1 \le i \le m)$, we denote by H_u the free product of the groups of the form

$$H(u_1 \dots u_{i-1} \cdot u_i \cdot u_{i+1} \dots u_m)$$

over all i from 1 to m. These groups are assigned to vertices of $\mathcal{K}(\mathcal{P})$. Now let us take an edge $e = (u, s \to t, v)$, where $u, v \in X^*$, $(s = t) \in \mathcal{S}$. The group $H_e = H_u \times H_v$ is assigned to it. The maps ι_e, τ_e naturally embed $H_e = H_u \times H_v$ into the groups $H_{usv} \cong H_u \times H_s \times H_v$ and $H_{utv} \cong H_u \times H_t \times H_v$, respectively (here H_u maps onto H_u, nd H_v maps onto H_v). It is clear that, instead of presenting edge groups of the form G_e, one can present an isomorphism of some subgroup of $H_{\iota(e)}$ to some subgroup of $H_{\tau(e)}$, for each egde e. In our case this isomorphism is very simple: it maps the subgroup $H_u \times \{1\} \times H_v \quad H_{usv}$ onto $H_u \times \{1\} \times H_v \quad H_{utv}$. In this case we will speak about the isomorphism induced by an edge e.

The groups of the form $H(U \cdot x \cdot V)$ will be presented as groups generated by edges of the form $U * f * V$, where f runs over the set of edges of the connected component of the Squier complex $\mathcal{K}(\hat{\mathcal{P}}_x)$ that contain vertex x. Here edges satisfy the relations $U * f * V = 1$ whenever f belongs to the tree $\mathcal{T}_x$, and also relations $U * r * V = 1$, where r is the defining path of a 2-cell of this complex. These relations of the group G were obtained in subsection c). For the direct products of the groups of the form $H(U \cdot x \cdot V)$, we introduce relations of commutativity: each element in the group $H(U \cdot x \cdot ZyV)$ commutes with each element in the group $H(UxZ \cdot y \cdot V)$. For the group G, such relations were obtained in subsection b).

Let us have an edge $e = (u, s \to t, v)$ that belongs to the complex $\mathcal{K}_w$. Let $x, y \in X$, $u = u_1xu_2$, $v = v_1yv_2$. The isomorphism induced by the edge e, takes $H(usv_1 \cdot y \cdot v_2)$ to $H(utv_1 \cdot y \cdot v_2)$, and it takes $H(u_1 \cdot x \cdot u_2yv)$ to $H(u_1 \cdot x \cdot u_2yv)$. So, according to (5, the conjugation by the edge e leads to the following relations

$$(usv_1 * f * v_2)^e = (utv_1 * f * v_2), \tag{12}$$

$$(u_1 * f * u_2sv)^e = (u_1 * f * u_2tv), \tag{13}$$

where f runs over the set of edges that generate the corresponding group in each of the cases. These relations coincide with relations (10) and (11) of the group G from subsection d). Finally, we represent the group $\pi_1(\mathcal{K}(\mathcal{Q}), w)$ as a group generated by edges of $\mathcal{K}_w(\mathcal{Q})$, claiming that the edges in $\mathcal{T}_Q$ are equal to the identity and adding relations that correspond to the defining paths of 2-cells of this complex. Such relations of the group G are described in subsection a). Thus the quotient group of the free product of the group $\pi_1(\mathcal{K}(\mathcal{Q}), w)$ and groups of the form H_u for all vertices u of $\mathcal{K}_w(\mathcal{Q})$, by the normal closure of relations (12) and (13), is given by the same generators and defining relations as G. This means that the diagram product $\mathcal{D}(G_X; \mathcal{S}, w)$ of the family $G_X = \{ G_x \ (x \in X) \}$ of groups over the presentation $\mathcal{Q} = \langle X \mid \mathcal{S} \rangle$ with base w is isomorphic to the group $G = \pi_1(\mathcal{K}(\mathcal{P}, w)$, that is, to the diagram group $\mathcal{D}(\mathcal{P}, w)$.

The Theorem is proved.

Now let us consider a few applications of Theorem 4. The first three of them deal with already known constructions. We give them to demonstrate that all group-theoretical constructions for diagram groups we dealt with earlier (see [12, Section 8]) are examples of diagram products. Then we show that the class of diagram groups is closed under some new operations: countable direct powers, restricted wreath products with the group $\mathbf{Z}$, and also under some new special construction that will be used in Section 5.

Example 5 Let $X = \{ x_1, \ldots, x_n \}$ be a finite alphabet. Consider the presentation $\mathcal{Q} = \langle X \mid \emptyset \rangle$ with empty set of defining relations and let $w = x_1 \ldots x_n$. To each letter x_i, we assign an arbitrary group G_i $(1 \le i \le n)$. It is obvious that the connected component of the Squier complex of $\mathcal{Q}$, which contains w, consists of exactly one vertex w. In the corresponding graph of groups, we have the group $G_w = G_1 \times \cdots \times G_n$. Obviously, it is the fundamental group of the 2-complex of groups from the definition of a diagram product. Thus the diagram product $\mathcal{D}(G_X; \mathcal{S}, w)$ of the family $G_X = \{ G_i \ (1 \le i \le n) \}$ of groups over the presentation $\mathcal{Q}$ with base w is the direct product $G_1 \times \cdots \times G_n$.

Example 6 Let I be a nonempty set and let G_i $(i \in I)$ be an arbitrary family of groups. Let us consider an alphabet $X = \{ x \} \cup \{ x_i \ (i \in I) \}$ and let $\mathcal{Q} = \langle X \mid \mathcal{S} \rangle$, where $\mathcal{S}$ consists of relations of the form $x = x_i$ for all $i \in I$. Let G_X be a family of groups that assigns the trivial group to the letter x and the group G_i to the letter x_i $(i \in I)$. The connected component of the Squier complex $\mathcal{K}(\mathcal{Q})$ containing x is a tree in which the vertex x is connected by edges with all vertices labelled by x_i $(i \in I)$. Let us consider the structure of a graph of groups on the 1-skeleton of the connected component of this Squier complex according to the definition 3. It is easy to see that all edge groups are trivial. From this, using description (4), it is easy to see that the fundamental group of the resulting 2-complex of groups is the free product of groups G_i $(i \in I)$. So the diagram product $\mathcal{D}(G_X; \mathcal{S}, x)$ of the family $G_X = \{ G_i \ (i \in I) \}$ of groups over the presentation $\mathcal{Q}$ with base x is the free product $* G_i$, $i \in I$.

Example 7 Let G, H be any groups. Let us consider the presentation $\mathcal{Q} = \langle X \mid \mathcal{S} \rangle$, where $X = \{ x, y, z \}$, $\mathcal{S} = \{ x = xy, z = yz \}$. Let $G_x = G$, $G_y = 1$, $G_z = H$ and consider the diagram product $\mathcal{D}(G_X; \mathcal{S}, xz)$ of the family $G_X = \{ G_x, G_y, G_z \}$ of groups over the presentation $\mathcal{Q}$ with base xz. We obtain that it is isomorphic to the group $G \bullet H$, where $\bullet$

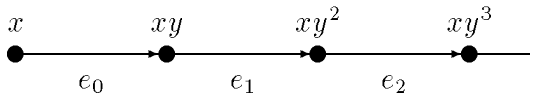
is the operation defined in [12]. One can check this directly by comparing the Theorem 4 and the definition of the operation $\bullet$ in [12]. Indeed, the group $G \bullet H$ can be described in the following way. Consider countable number of copies G_i of the group G and countable number of copies H_i of the group H $(i \in \mathbf{Z})$. An infinite cyclic group $\langle\, z \,\rangle$ acts on the group

$$(* G_i) \times (* H_i) \tag{14}$$

(free products are taken over all $i \in \mathbf{Z}$) permuting the factors: it takes G_i to G_{i+1} and H_i to H_{i+1} for all integers i. The group $G \bullet H$ is the semidirect product of the group (14) and the group $\langle\, z \,\rangle$.

Example 8 Let G be an arbitrary group. Let us consider the presentation $\mathcal{Q} = \langle\, X \mid \mathcal{S} \,\rangle$, where $X = \{\, x, y \,\}$, $\mathcal{S} = \{\, x = xy \,\}$. Let $G_x = 1$, $G_y = G$. Consider the diagram product $\mathcal{D}(G_X; \mathcal{S}, x)$ of the family $G_X = \{\, G_x, G_y \,\}$ of groups over the presentation $\mathcal{Q}$ with base x. Let us show that it is isomorphic to the countable direct power of the group G.

The connected component $\mathcal{K}_x$ of the Squier complex over $\mathcal{Q}$ containing x has the following form:



Here vertices are all words of the form xy^i $(i \geq 0)$, positive edges have the form $e_i = (1, x \to xy, y^i)$ $(i \geq 0)$, and the maximal subtree $\mathcal{T}$ includes all these edges. This complex has no 2-cells. Thus it is obvious that its fundamental group is trivial. Using our convention that (given a maximal subtree) all edges are regarded as elements of the fundamental groups, we have equalities $e_i = 1$ for all $i \geq 0$.

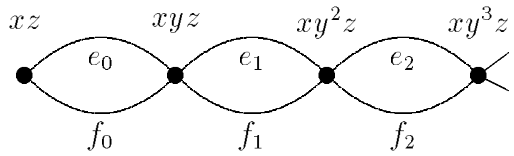
Let us consider the structure of the graph of groups on the 1-skeleton of the complex $\mathcal{K}_x$ according to the definition of a diagram product. We will obtain that the group $H_v = G_{n1} \times \cdots \times G_{nn}$, where G_{ni} $(1 \leq i \leq n)$ are groups isomorphic to G, corresponds to the vertex $v = xy^n$ $(n \geq 0)$. Let us consider a positive edge $e = e_n = (1, x \to xy, y^n)$ $(n \geq 0)$. The group G_e is isomorphic to G^n, the nth direct power of G. The embedding ι_e maps G^n isomorphically onto $G_{n1} \times \cdots \times G_{nn}$, and the mapping τ_e maps G^n isomorphically onto the last n factors of the direct product $G_{n+1,1} \times G_{n+1,2} \cdots \times G_{n+1,n+1}$. The relations from the description (5), together with the equality $e = 1$, allow to identify corresponding elements of G_{n1} and $G_{n+1,2}$, …, G_{nn} and $G_{n+1,n+1}$. Thus we can introduce the following notation: $G_0 = G_{11} = G_{22} = \cdots$, $G_1 = G_{21} = G_{32} = \cdots$, …, $G_n = G_{n+1,1} = G_{n+2,2} = \cdots$, … . Each of these groups is isomorphic to G. They generate a countable direct power of G. So our diagram product is the countable direct power of G.

Theorem 4 implies the following result.

Theorem 9 *The class of diagram groups is closed under countable direct powers.*

Example 10 Let G be arbitrary group. Consider the presentation $\mathcal{Q} = \langle X \mid \mathcal{S} \rangle$, where $X = \{ x, y, z \}$, $\mathcal{S} = \{ x = xy, z = yz \}$. Let $G_x = 1$, $G_y = G$, $G_z = 1$ and consider the diagram product $\mathcal{D}(G_X; \mathcal{S}, xz)$ of the family $G_X = \{ G_x, G_y, G_z \}$ of groups over the presentation $\mathcal{Q}$ with the base xz. Let us show that it is isomorphic to the (restricted) wreath product G wr $\mathbf{Z}$.

The connected component $\mathcal{K}_{xz}$ of the Squier complex over $\mathcal{Q}$ that contains xz has the following form:



Here vertices are all words of the form xy^iz $(i \geq 0)$, positive edges have the form $e_i = (1, x \to xy, y^iz)$, $f_i = (xy^i, z \to yz, 1)$ $(i \geq 0)$, and the maximal subtree $\mathcal{T}$ consists of the edges e_i, $i \geq 0$. All 2-cells can be described as follows. Let $i \geq 0$. Consider the vertex xy^iz. The edges e_i, f_i going out of it correspond to independent transformations of words. So the given pair of edges defines two homotopic paths $e_i f_{i+1}$ and $f_i e_{i+1}$ that define a 2-cell.

According to the convention that edges are regarded as elements of the fundamental group $\pi_1(\mathcal{K}, xz)$, we have $e_i = 1$ $(i \geq 0)$ in the group. The equalities $e_i f_{i+1} = f_i e_{i+1}$ that hold in this group imply $f_i = f_{i+1}$ for all $i \geq 0$. Let $f = f_0 = f_1 = f_2 = \cdots$.

According to the definition of a diagram product, let us consider the structure of the graph of groups on the 1-skeleton of $\mathcal{K}_{xz}$. The group $H_v = G_{n1} \times \cdots \times G_{nn}$ is assigned to the vertex $v = xy^nz$ $(n \geq 0)$, where G_{ni} $(1 \leq i \leq n)$ is a group isomorphic to G. Let us consider relations (5) that correspond to positive edges. Let $e = e_n = (1, x \to xy, y^n)$ $(n \geq 0)$. As in the previous example, using the equality $e = 1$, we identify corresponding elements of the groups G_{ni} and $G_{n+1,i+1}$ $(1 \leq i \leq n)$ and introduce the notation $G_0 = G_{11} = G_{22} = \cdots$, $G_1 = G_{21} = G_{32} = \cdots$, $\ldots$, $G_n = G_{n+1,1} = G_{n+2,2} = \cdots$, $\ldots$. As above, these groups generate a countable direct power of the group G. Now let $e = f_n = (xy^n, z \to yz, 1)$ $(n \geq 0)$. Consider relations of the form (5) that correspond to these edges. The group G_e is still the nth power of G. The embedding ι_e maps G^n onto $G_{n1} \times \cdots \times G_{nn}$ isomorphically, and the embedding τ_e maps G^n isomorphically onto the first n factors of the direct product $G_{n+1,1} \times G_{n+1,2} \cdots \times G_{n+1,n+1}$. So the relations that correspond to the edge f show that conjugation by f takes G_{ni} to $G_{n+1,i}$ $(1 \leq i \leq n)$. Using our notation, we obtain that the conjugation by f takes the group G_k to the group G_{k+1} for all $k \geq 0$. Thus the diagram product we are considering is generated by groups G_0, G_1, $\ldots$ and the element f. From this, one can deduce that we have the restricted wreath product G wr $\mathbf{Z}$.

Applying Theorem 4, we get one more result.

Theorem 11 *The class of diagram groups is closed under restricted wreath products with the infinite cyclic group* $\mathbf{Z}$, *that is, if* G *is a diagram group, then* G wr $\mathbf{Z}$ *is also a diagram group.*

Note that if we take R. Thompson's group F represented by diagrams over $\langle\, u \mid uu = u\,\rangle$ with base u and consider the presentation $\langle\, x, u, z \mid xu = x, uz = z, uu = u\,\rangle$, then the diagram group over it with base xz will be isomorphic not to F wr $\mathbf{Z}$ but to F. This can be checked directly. To get the group F wr $\mathbf{Z}$, one needs to represent the group F by diagrams according to the statement of Theorem 4. Namely, one has to take the diagram group with base y over the presentation $\langle\, u, a, y \mid y = aua, uu = u\,\rangle$. Then the diagram group with base xz over $\mathcal{P} = \langle\, x, y, z, a, u \mid xy = x, yz = z, y = aua, uu = u\,\rangle$ will be isomorphic to F wr $\mathbf{Z}$. The reader can easily list the presentations that lead to diagram groups of the form $(\cdots((\mathbf{Z} \text{ wr } \mathbf{Z}) \text{ wr } \mathbf{Z})\cdots) \text{ wr } \mathbf{Z}$.

Let us make one more remark. In [12] we constructed an example of a diagram group that was finitely generated but not finitely presented (Theorem 10.5). We took the group $\mathbf{Z} \bullet \mathbf{Z}$ for this purpose. It has a presentation with three generators

$$\mathbf{Z} \bullet \mathbf{Z} = \langle\, a, b, t \mid [a^{t^n}, b] = 1 \ (n \geq 0)\,\rangle.$$

Now we can also take the group $\mathbf{Z}$ wr $\mathbf{Z}$ as an example of a finitely generated but not finitely presented group (the fact that $\mathbf{Z}$ wr $\mathbf{Z}$ has no finite presentations can be easily proved using either HNN-extensions or representation of groups by transformations). We have the following presentation with two generators for this group:

$$\mathbf{Z} \text{ wr } \mathbf{Z} = \langle\, a, b \mid [a^{b^n}, a] = 1 \ (n \geq 1)\,\rangle.$$

In the next example we deal with a more complicated construction. At first sight one can think it is quite artificial. However, we will efficiently use it later, in Section 5. Let us consider the following group-theoretical construction. Take two groups, G and H. We assign to them a new group denoted by $\mathcal{O}(G, H)$. Let us consider a countable family of copies G_i of the group G, and a coutable family of copies H_i of H ($i \in \mathbf{Z}$). For any $i \in \mathbf{Z}$, let g_i (h_i) be the element that corresponds to $g \in G$ ($h \in H$). By G^∞ (H^∞) we denote a coutable direct power of the groups G_i (H_i) taken over all $i \in \mathbf{Z}$. Let

$$\mathcal{O}(G, H) = G^\infty * H^\infty * \langle\, c\,\rangle / \mathcal{N}, \tag{15}$$

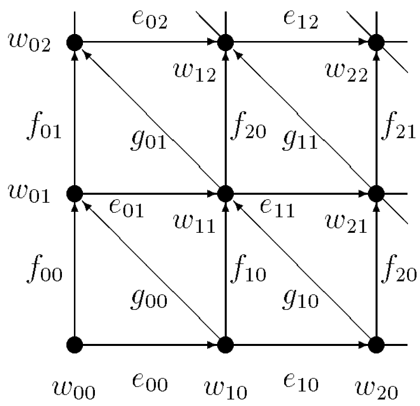
where $\mathcal{N}$ is the normal closure of the set of relations of the two forms:

$$g_i^t = g_{i+1}, \quad h_i^t = h_{i+1} \quad \text{for all } i \in \mathbf{Z},\ g \in G,\ h \in H; \tag{16}$$

$$[g_i, h_j] = 1 \ \text{ for all } i, j \in \mathbf{Z},\ g \in G,\ h \in H \text{ such that } i \leq j. \tag{17}$$

Example 12 Let G, H be arbitrary groups. Let us consider the presentation $\mathcal{Q} = \langle\, X \mid \mathcal{S}\,\rangle$, where $X = \{\, x, y, \bar{y}, z, p, q, r\,\}$, $\mathcal{S} = \{\, x = xyp, z = r\bar{y}z, pyq = q\bar{y}r\,\}$. Let $G_y = G$, $G_{\bar{y}} = H$, $G_x = G_z = G_p = G_q = G_r = 1$. Consider the diagram product $\mathcal{D}(G_X; \mathcal{S}, w)$ of the family $G_X = \{\, G_x, G_y, G_{\bar{y}}, G_z, G_p, G_q, G_r\,\}$ of groups over the presentation $\mathcal{Q}$ with base $w = xyq\bar{y}z$. Let us show that it is isomorphic to $\mathcal{O}(G, H)$.

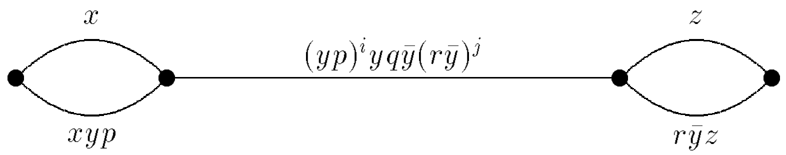
The connected component $\mathcal{K}_w$ of the Squier complex over $\mathcal{Q}$ that contains w, has the following form:



Here the vertices are words $w_{ij} = x(yp)^i yq\bar{y}(r\bar{y})^j z$ $(i, j \geq 0)$. The positive edges have the form $e_{ij} = (1, x \to xyp, (yp)^i yq\bar{y}(r\bar{y})^j z)$, $f_{ij} = (x(yp)^i yq\bar{y}(r\bar{y})^j, z \to r\bar{y}z, 1)$ and $g_{ij} = (x(yp)^i y, pyq \to q\bar{y}r, \bar{y}(r\bar{y})^j z)$ $(i, j \geq 0)$. We choose the maximal subtree $\mathcal{T}$ formed by the edges e_{ij} for all $i, j \geq 0$ and also by the edges f_{0j} for $j \geq 0$. Thus our convention that the choice of $\mathcal{T}$ makes the edges to be elements of the fundamental group $\pi_1(\mathcal{K}, w)$, leads to the equalities $e_{ij} = 1$ $(i, j \geq 0)$, $f_{0j} = 1$ $(j \geq 0)$ in the fundamental group of the complex.

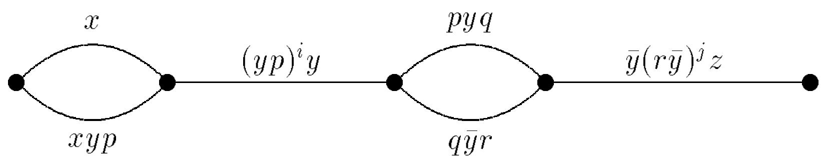
Let us describe all 2-cells of the complex $\mathcal{K}_w$. Recall that there are defining relations of three types in $\mathcal{Q}$: $x = xyp$, $pyq = q\bar{y}r$, $z = r\bar{y}z$. If we have two independent applications of elementary transformations to words of the form w_{ij}, then it is easy to see that they belong to different types because each of the letters x, q, z occurs into the word w_{ij} only once. Therefore, we have exactly three situations.

1) The relations applied in the independent transformations are $x = xyp$ and $z = r\bar{y}z$ (see the picture below).



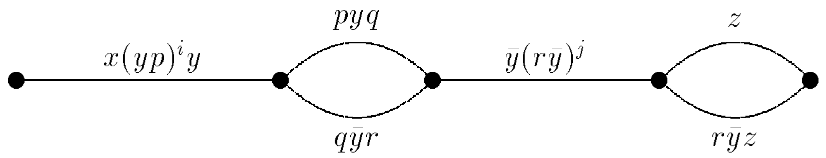
This diagram corresponds to the two paths in the Squier complex: $e_{ij}f_{i+1,j}$ and $f_{ij}e_{i,j+1}$. This leads to relations $e_{ij}f_{i+1,j} = f_{ij}e_{i,j+1}$. Simplifying, we have $f_{i+1,j} = f_{ij}$ for all $i, j \geq 0$. It is obvious that f_{ij} does not depend on i. So $f_{0j} = 1$ gives $f_{ij} = 1$ for all $i, j \geq 0$.

2) The relations are $x = xyp$ and $pyq = q\bar{y}r$ (see the picture below).



In this case, we have the equality $e_{i+1,j}g_{i+1,j} = g_{i,j}e_{i,j+1}$, that is, $g_{i+1,j} = g_{ij}$ for all $i, j \geq 0$. This means that g_{ij} does not depend on i.

3) The relations are $pyq = q\bar{y}r$ and $z = r\bar{y}z$ (see the picture below).



Here we have the equality $g_{i,j}f_{i,j+1} = f_{i+1,j}g_{i,j+1}$. Hence $g_{ij} = g_{i,j+1}$ for all $i, j \geq 0$. Therefore, g_{ij} depends on neither i nor j. For convenience, let $c = g_{ij}$ for all $i, j \geq 0$.

Let us take an arbitrary vertex $v = w_{ij} = x(yp)^iyq\bar{y}(r\bar{y})^jz$ for some $i, j \geq 0$. In the graph of groups that corresponds to the diagram product, the product of $(i+1)$th power of G and the $(j+1)$th power of H will correspond to the vertex $G_{w_{ij}}$ (the number of factors is just the number of occurrences of y and $\bar{y}$ in v, respectively). Thus we can present the group $G_{w_{ij}}$ in the form

$$K_{ij} = L_{iji} \times \cdots L_{ij0} \times H_{ij0} \times \cdots H_{ijj},$$

where the factors of the form L_{ijk} are isomorphic to G, and the factors of the form H_{ijk} are isomorphic to H. Relations (5) that correspond to a positive edge $e = (u, s \to t, v)$ will be studied with respect to the type of the involved defining relation $(s = t) \in \mathcal{S}$ (there are three types of them).

1) $s = x$, $t = xyp$. We have $e = (1, x \to xyp, v)$, where $v = (yp)^iyq\bar{y}(r\bar{y})^jz$ for some $i, j \geq 0$. The group G_e is isomorphic to $G^{i+1} \times H^{j+1}$. It maps isomorphically onto K_{ij} under ι_e. Note that $K_{i+1,j}$ is the direct product of $L_{i+1,j,i+1}$ and the isomorphic image of K_{ij} under τ_e. Using the fact that $e = e_{ij} = 1$ we see that relation (5) identifies some subgroups. Let us write down these identifications as equalities. By these equalities we mean that the corresponding elements of equal groups are identified. We have: $L_{ijk} = L_{i+1,j,k}$ for $0 \leq k \leq i$ and $H_{ijk} = H_{i+1,j,k}$ for $0 \leq k \leq j$.

2) $s = z$, $t = r\bar{y}z$. Now $e = (u, z \to r\bar{y}z, 1)$, where $u = x(yp)^iyq\bar{y}(r\bar{y})^j$ for some $i, j \geq 0$. Arguing analogously to the previous case and taking into account that $e = f_{ij} = 1$, we conclude that relation (5) leads to the following identifications of subgroups: $L_{ijk} = L_{i,j+1,k}$ for $0 \leq k \leq i$ and $H_{ijk} = H_{i,j+1,k}$ for $0 \leq k \leq j$.

Summarizing what we got in the first two cases of relations, we see that groups L_{ijk}, H_{ijk} depend of k only. In other words, one can introduce groups L_k, H_k $(k \geq 0)$ in such a way that the equalities $L_{ijk} = L_k$ for all $i \geq k$, $j \geq 0$, and $H_{ijk} = H_k$ for all $i \geq 0$, $j \geq k$ hold in our diagram product.

3) $s = pyq$, $t = q\bar{y}r$. In this case $e = (u, pyq \to q\bar{y}r, v)$, where $u = x(yp)^iy$, $v = \bar{y}(r\bar{y})^j$ for some $i, j \geq 0$. In the fundamental group $\pi_1(\mathcal{K}, w)$, the equality $e = g_{ij} = c$ holds, as it was shown above, the group G_e is isomorphic to $G^{i+1} \times H^{j+1}$. We have

$$G_{\iota(e)} = L_{i+1} \times \cdots \times L_0 \times H_0 \times \cdots \times H_j,$$

ι_e maps G^{i+1} onto the direct product $L_{i+1} \times \cdots \times L_1$ and it maps H^{j+1} onto the direct product $H_0 \times \cdots \times H_j$. Analogously,

$$G_{\tau(e)} = L_i \times \cdots \times L_0 \times H_0 \times \cdots \times H_{j+1},$$

τ_e maps G^{i+1} onto the direct product $L_i \times \cdots \times L_0$ and it maps H^{j+1} onto the direct product $H_1 \times \cdots \times H_{j+1}$. Therefore, conjugating by the element c takes $L_{i+1}, \ldots, L_1$ to L_i, $\ldots$, L_0 respectively. The subgroups $H_0, \ldots, H_j$ are taken to $H_1, \ldots, H_{j+1}$ respectively, under this conjugation. Briefly, we can write $L_{i+1}^c = L_i$, $H_j^c = H_{j+1}$ for any $i, j \geq 0$.

Thus the equalities $L_i = L_0^{c^{-i}}$, $H_j = H_0^{c^j}$ hold for any nonnegative integers i, j. Let us extend these equalities to the case of negative i, j regarding these equalities as definitions. Note that elements from different subgroups of the form L_i ($i \geq 0$) commute. So the analogous fact is true for all integers i. The same fact is true for subgroups H_j for all $j \in \mathbf{Z}$. Let $G_i = L_{-i}$ ($i \in \mathbf{Z}$) by definition. Obviously, $G_i^c = G_{i+1}$, $H_i^c = H_{i+1}$ for all $i \in \mathbf{Z}$. This means that relations (16) hold. We also have conditions that any element in L_0, L_1, $\ldots$ commutes with any element in H_0, H_1, $\ldots$. In particular, $[L_0, H_{j-i}] = 1$ for any $j \geq i$. Taking into account that $G_0 = L_0$ and conjugating by the element c^i, we obtain $[G_i, H_j] = 1$ for $i \leq j$, that is, relations (17) hold. It is easy to see that these relations are in fact equivalent to the condition that $[L_i, H_j] = 1$ for any $i, j \geq 0$. Indeed, the inequality $-i \leq j$ and relations (16) allow us to conclude that $[G_{-i}, H_j] = 1$, where G_{-i} is L_i.

Thus we see that the diagram product we have calculated is in fact the group given by relations (16) and (17), that is, it is isomorphic to $\mathcal{O}(G, H)$.

Using Theorem 4, we have the following result.

Theorem 13 *If G, H are diagram groups, then $\mathcal{O}(G, H)$ is also a diagram group.*

The previous example shows in details how, given two diagram groups G and H, one can construct a presentation and a base, for which $\mathcal{O}(G, H)$ will be a diagram group.

3 Nilpotent and Abelian Subgroups of Diagram Groups

We know from the previous section that soluble subgroups of any degree can be subgroups of diagram groups. Contrary to that, we shall prove in this Section that any nilpotent subgroup of a diagram group is abelian. We will also establish the fact that all abelian subgroups of diagram groups are free abelian. This will generalize the result that any abelian diagram group is free abelian. Finally, we shall describe finite sets of pairwise commuting diagrams, generalizing a description of pairs of commuting diagrams from [12].

We will use some concepts from combinatorics on diagrams from [12, Section 15]. For reader's convenience, let us recall some definitions.

A spherical diagram is called *absolutely reduced* if any positive integer power of it is reduced (does not contain dipoles). A spherical diagram is called *normal* if it cannot be decomposed into a sum of two non-spherical diagrams. We proved [12, Theorem 15.14] that for any spherical diagram Δ there exists an absolutely reduced normal spherical diagram $\hat{\Delta}$ (that may have different base, in general) and some (not necessarily spherical) diagram Ψ such that $\Delta = \Psi^{-1}\hat{\Delta}\Psi$.

Theorem 14 *Let H be an arbitrary subgroup of a diagram group $\mathcal{D}(\mathcal{P}, w)$. Then the centre of H and the commutator subgroup of H intersect trivially that is, $Z(H) \cap H' = 1$.*

Proof. Let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group and let H be a subgroup of G. Suppose that $Z(H) \cap H' \neq 1$. Consider a nontrivial element $g \in Z(H) \cap H'$ and let Δ be a diagram representing it. Applying [12, Lemma 15.10c], we find an absolutely reduced diagram Δ_0 that is conjugated to Δ. Let $\Delta_0 = \Psi^{-1}\Delta\Psi$, where Ψ is a (w, w_0)-diagram. Conjugation by Ψ is an isomorphism that takes the group G to the group $G_0 = \mathcal{D}(\mathcal{P}, w_0)$. Under this isomorphism, the subgroup H is taken to a subgroup H_0, where $g_0 \in Z(H_0) \cap H_0'$, and the element g_0 is represented by an absolutely reduced (w_0, w_0)-diagram Δ_0.

Let us decompose the diagram Δ_0 into a sum of components: $\Delta_0 = A_1 + \cdots + A_m$, where A_i is a spherical (w_i, w_i)-diagram $(1 \leq i \leq m)$. As in [12, Theorem 15.35], we conclude that the centralizer of g_0 is the direct sum of centralizers of the elements represented by diagrams $A_1, \ldots, A_m$. More precisely, if Γ is a spherical (w_i, w_i)-diagram that commutes with Δ_0 in the group G_0, then $\Gamma = B_1 + \cdots + B_m$, where B_i is a (w_i, w_i)-diagram that commutes with A_i. By the assumption, any diagram representing an element in H_0, commutes with Δ_0 since g_0 belongs to the centre of H_0.

Since Δ_0 represents a nontrivial element, there exists an integer i between 1 and m such that the diagram A_i is nontrivial and so it is a simple absolutely reduced diagram. Its centralizer is cyclic (see the proof of Theorem 15.35 in [12]). Let us now take two diagrams Γ, Ξ that represent elements in H_0. By the arguments of the above paragraph, there are decompositions of the form $\Gamma = B_1 + \cdots + B_m$, $\Xi = C_1 + \cdots + C_m$, where B_i, C_i are (w_i, w_i)-diagrams that commute with A_i. Cyclicity of the centralizer of A_i implies that B_i and C_i commute, that is, $[B_i, C_i] = \varepsilon(w_i)$. Therefore $[\Gamma, \Xi] = [B_1, C_1] + \cdots + [B_m, C_m] = \Delta' + \varepsilon(w_i) + \Delta''$, where Δ', Δ'' are spherical diagrams with bases $w_1 \ldots w_{i-1}$, $w_{i+1} \ldots w_m$. It is clear that the product of diagrams of the form $\Delta' + \varepsilon(w_i) + \Delta''$ is again a diagram of the same form. Hence any element of the commutator subgroup of the group H_0 has the form $\Delta' + \varepsilon(w_i) + \Delta''$. This contradicts the condition $\Delta_0 = A_1 + \cdots + A_m$, where $A_i \neq \varepsilon(w_i)$.

The Theorem is proved.

Corollary 15 *Any nilpotent subgroup of a diagram group is abelian.*

Proof. Let $\mathcal{D}(\mathcal{P}, w)$ be a diagram group and let K be its nilpotent subgroup. If K is not abelian then K has a (non-abelian) nilpotent subgroup H of degree 2. This means that the commutator subgroup of H is contained in its centre, that is, $H' \subseteq Z(H)$. Theorem 14 claims that the centre of H and its commutator subgroup have trivial intersection so $H' = 1$, that is, H is abelian, a contradiction.

Let us now describe all abelian subgroups of diagram groups. It turns out that all of them are free abelian. Note that if there was an abelian subgroup in a diagram group that was not free abelian, then we would immediately disprove the Subgroup Conjecture, because it is known from [12] that the quotient of any diagram group by its commutator subgroup is free abelian.

Theorem 16 *Any abelian subgroup of a diagram group is free abelian.*

Proof. Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation, let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group and let $H \leq G$ be an abelian subgroup of G. If $H = 1$ then we have nothing to prove. Let $H \neq 1$. Consider an element $h \in H$, $h \neq 1$. Using certain conjugation and replacing the subgroup by an isomorphic one, we can assume by [12, Lemma 15.14] that h is represented by an absolutely reduced normal diagram Δ that is decomposed into the sum of components: $\Delta = \Delta_1 + \cdots + \Delta_m$, where Δ_i is a spherical diagram with base u_i ($1 \leq i \leq m$). Since H is abelian, it is contained in the centralizer of h. Thus any element g in H decomposes into a sum of (u_i, u_i)-diagrams. Denote by $\psi_i(g)$ the ith summand of this decomposition. It is easy to see that ψ_i is a homomorphism of the group H into the diagram group over $\mathcal{P}$ with base u_i.

Let $1 \leq k \leq m$ be a number such that Δ_k is nontrivial. The centralizer of Δ_k is cyclic [12, Theorem 15.35]. Consider the homomorphism ψ_k. Firstly, $\psi_k(h) = \Delta_k \neq \varepsilon(u_i)$. Secondly, the image of ψ_k is contained in the centralizer of Δ_k, that is, in a cyclic group. We thus proved that for any $h \in H$, $h \neq 1$, there exists a homomorphism $\psi \colon H \to \mathbf{Z}$ such that $\psi(h) \neq 1$. This means that H is residually cyclic, that is, it embeds into a Cartesian power of the infinite cyclic group. An easy argument in spirit of linear algebra (using the Choice Axiom) shows that a Cartesian power $\mathbf{Z}$ is a free abelian group. Therefore, H is also free abelian.

In conclusion of this Section we give a simple but useful generalization of Theorem 15.34 from [12]. We will need it later in Section 6.

Theorem 17 *Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation and let $G = \mathcal{D}(\mathcal{P}, w)$ for some $w \in \Sigma^+$. Suppose that $A_1, \ldots, A_m$ are spherical diagrams with base w that pairwise commute in G. Then there exist a word $v = v_1 \ldots v_n$, spherical (v_j, v_j)-diagrams Δ_j ($1 \leq j \leq n$), integers d_{ij} ($1 \leq i \leq m$, $1 \leq j \leq n$) and some (w, v)-diagram Γ such that*

$$\Gamma^{-1} A_i \Gamma = \Delta_1^{d_{i1}} + \cdots + \Delta_n^{d_{in}}$$

for all $1 \leq i \leq m$. We can additionally assume that each of the diagrams $\Delta_1, \ldots, \Delta_n$ is either trivial or simple absolutely reduced.

Proof. First of all let us show that the additional assumption about Δ_i can be proved provided the main statement of the theorem is proved. If we have already found the decompositions of diagrams from the main statement of the Theorem, then by Lemma 15.14 from [12] we can find diagrams Ψ_j such that diagarams $\Psi_j \Delta_j \Psi_j^{-1}$ are normal and absolutely reduced. Each of these diagarams decomposes into a sum of components that are either trivial or simple absolutely reduced. After that we apply additional conjugation by $\Psi_1 + \cdots + \Psi_n$ replacing each of the n summands by a sum of components.

The rest will be proved by induction on m. If $m = 1$ or $m = 2$ then it is proved in [12] (Lemma 15.10c and Theorem 15.34). So we assume that $m > 2$, and the statement is true for all values less than m. Consider the diagram A_m. Applying Lemma 15.10c, we find a (w, u)-diagram Ψ such that the diagram $A'_m = \Psi^{-1} A_m \Psi$ will be absolutely reduced and normal. It can be decomposed into the sum of components: $A'_m = B_1 + \cdots + B_k$, where B_j is a (u_j, u_j)-diagram for some u_j $(1 \le j \le k)$ and $u = u_1 \ldots u_k$. Each of the diagrams B_j $(1 \le j \le k)$ is either trivial or simple and all of them are absolutely reduced and normal. Let $A'_i = \Psi^{-1} A_i \Psi$ for all $1 \le i \le m-1$. It is clear that all diagrams $A'_1, \ldots, A'_m$ pairwise commute. Since $A'_1, \ldots, A'_{m-1}$ also commute with A'_m, each of them can be decomposed into a sium of (u_j, u_j)-diagrams $(1 \le j \le n)$ by [12, Theorem 15.35]. We have $A'_i = C_{i1} + \cdots + C_{in}$, where $1 \le i \le m-1$, each of the diagrams C_{ij} is a (u_j, u_j)-diagram and C_{ij} commutes with B_j $(1 \le i \le m-1,\ 1 \le j \le n)$. Suppose that r $(1 \le r \le k)$ is a number such that the component B_r is nontrivial. Since B_r is a simple absolutely reduced diagram, it has a cyclic centralizer (see the proof of Theorem 15.35). In particular, there exists a spherical diagram Δ_r with base u_r such that any of the diagrams $C_{1r}, \ldots, C_{m-1,r}, B_r$ is a power of Δ_r. Let $C_{ir} = \Delta_r^{d_{ir}}$ $(1 \le i \le m-1)$, $B_r = \Delta_r^{d_{mr}}$. We do that with each r $(1 \le r \le k)$, for which the component B_r is nontrivial.

Now let $1 \le r \le k$ be such that the component B_r is trivial, that is, $B_r = \varepsilon(u_r)$. Applying the inductive assumption to diagrams $C_{1r}, \ldots, C_{m-1,r}$, we find a word $v_r = v_{1r} \ldots v_{n_r,r}$, spherical (v_j, v_j)-diagrams Δ_{jr} $(1 \le j \le n_r)$, integers d_{ijr} $(1 \le i \le m-1$, $1 \le j \le n_r)$ and some (u_r, v_r)-diagram Γ_r such that

$$\Gamma_r^{-1} C_{ir} \Gamma_r = \Delta_{1r}^{d_{i1r}} + \cdots + \Delta_{n_r,r}^{d_{i,n_r,r}} \tag{18}$$

for all $1 \le i \le m-1$. Now $\Gamma_r^{-1} B_r \Gamma_r = \varepsilon(v_r) = \varepsilon(v_{1r}) + \cdots + \varepsilon(v_{n_r,r})$, and one can put $d_{m1r} = \cdots = d_{m,n_r,r} = 0$. Then equality (18) will be true also for $i = m$, if we put $C_{mr} = B_r$.

For numbers r such that B_r is nontrivial we put $v_r = u_r$, and take the trivial diagram for Γ_r. In this case, we also need to put $n_r = 1$, $\Delta_{1r} = \Delta_r$, $d_{i1r} = d_{ir}$. Then the equalities (18) are true for all $1 \le r \le k$, $1 \le i \le m$. Putting $\Gamma = \Psi(\Gamma_1 + \cdots + \Gamma_k)$, we see that

$$\Gamma^{-1} A_i \Gamma = \sum_{r=1}^{k} \sum_{j=1}^{n_r} \Delta_{jr}^{d_{ijr}},$$

for $1 \le i \le m$, that is, we have got the required decomposition of diagrams into a sum.

The proof is complete.

In [12] we established that the conjugacy problem is decidable for any diagram group $\mathcal{D}(\mathcal{P}, w)$, where $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is a semigroup presentation with decidable word problem. In particular, this implies the decidability of the conjugacy problem in R. Thompson's group F. We can pose a more general problem — a uniform conjugacy problem for sequences.

Problem 1 *Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation with decidable word problem, $w \in \Sigma^+$, $G = \mathcal{D}(\mathcal{P}, w)$. Does there exist an algorithm that decides, given two sequences of elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of the group G (elements are represented by diagrams), whether there is an element $z \in G$ such that $x_i^z = y_i$ for all i from 1 to n? In particular, is this problem decidable for R. Thompson's group F?*

Note that there is some analogy between diagram groups and matrix groups (we remarked about this in [12]). The corresponding question for matrix groups was solved positively in [29] and independently in [9].

4 Soluble Subgroups in Diagram Groups

In this section we shed some light on the structure of soluble subgroups in diagram groups. First of all let us consider an example that demonstrates that there exist soluble subgroups of any degree in R. Thompson's group F. Let $\mathcal{P} = \{x \mid xx = x\}$. All groups $\mathcal{D}(\mathcal{P}, x^k)$, where $k = 1, 2, \ldots$, are isomorphic to F. Consider any nontrivial (x, x)-diagram Δ. Then the diagrams $\Delta_1 = \varepsilon(x^2) + \Delta + \varepsilon(x)$ and $\Delta_2 = \varepsilon(x) + \Delta + \varepsilon(x^2)$ are conjugate because they are sums of components that conjugate respectively. Secondly, the diagrams commute which can be seen directly and are conjugated by the diagram $\Gamma = (x^2 \to x) + \varepsilon(x) + (x \to x^2)$. Denoting by a, b the elements of $\mathcal{D}(\mathcal{P}, x^4)$ that represent diagrams Δ_1 and Γ, respectively, we can see that a and b generate the group $\mathbf{Z}$ wr $\mathbf{Z}$, a (restricted) wreath product of two infinite cyclic groups. In the following theorem we present a more general form of the above construction.

Theorem 18 *Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation. Suppose that there exist nonempty words x, y, z over Σ such that $xy = x$, $yz = z$ modulo $\mathcal{P}$, and suppose that the diagram group $\mathcal{D}(\mathcal{P}, y)$ is nontrivial. Then the group $G = \mathcal{D}(\mathcal{P}, xyz)$ contains a subgroup isomorphic to $\mathbf{Z}$ wr $\mathbf{Z}$. Namely, let Δ be any nontrivial (y, y)-diagram, let Γ_1 be arbitrary (xy, x)-diagram, and let Γ_2 be arbitrary (z, yz)-diagram. Then elements a and b, represented by diagrams $\varepsilon(x) + \Delta + \varepsilon(z)$ and $\Gamma_1 + \Gamma_2$, respectively, generate in G a subgroup isomorphic to $\mathbf{Z}$ wr $\mathbf{Z}$.*

Proof. First of all, let us mention that elements xz, xyz, $xy^2z, \ldots$ are equal modulo $\mathcal{P}$ so the diagram groups with these bases over $\mathcal{P}$ will be isomorphic to each other. To get the diagrams from the above example, one needs to put $x = y = z$ and then go from the group with base $xyz = x^3$ to the group with base $xy^2z = x^4$ using conjugation by the element $(x \to x^2) + \varepsilon(x^2)$.

To show that elements a, b of some group G generate $\mathbf{Z}$ wr $\mathbf{Z}$, it suffices to show that the elements $a_i = a^{b^i}$ $(i \in \mathbf{Z})$ form a free basis of a free abelian group. To check this, it suffices to show that for any positive integer n, the elements $a_0, a_1, \ldots, a_n$ form a basis of the free abelian group they generate. We will explicitly find the elements a_i $(0 \le i \le n)$ of the corresponding diagram group $\mathcal{D}(\mathcal{P}, xyz)$. For convenience, we will go to the diagram group over $\mathcal{P}$ with base $xy^{n+1}z$ using conjugation by the diagram

$$\Psi = (\Gamma_1^{-1} + \varepsilon(yz))(\Gamma_1^{-1} + \varepsilon(y^2z)) \cdots (\Gamma_1^{-1} + \varepsilon(y^nz)).$$

One can check directly that $c_i = \Psi^{-1}a_i\Psi = \varepsilon(xy^{n-i}) + \Delta + \varepsilon(y^iz)$ for all $0 \le i \le n$. It is clear that the elements c_0, c_1, ..., c_n pairwise commute. The obvious formula

$$c_0^{d_0}c_1^{d_1}\dots c_n^{d_n} = \varepsilon(x) + \Delta^{d_0} + \Delta^{d_1} + \dots + \Delta^{d_n} + \varepsilon(z)$$

shows that the elements c_0, c_1, ..., c_n form a basis of the free abelian subgroup in $\mathcal{D}(\mathcal{P}, xy^{n+1}z)$. So the elements a_0, a_1, ..., a_n also form a basis of a free abelian subgroup of $\mathcal{D}(\mathcal{P}, xyz)$ as desired.

We will return to the group $\mathbf{Z}$ wr $\mathbf{Z}$ later. Now we shall prove a simple fact that will imply that F contains soluble subgroups of any degree.

Lemma 19 *(Restricted) wreath product F* wr $\mathbf{Z}$ *is a subgroup of F.*

Proof. We will use some known properties of R. Thompson's group F mentioned in Section 1. As we mentioned above, there are several representations of F by piecewise linear functions. Let us consider the representation by functions on $[0, \infty)$. For any positive integer k we consider the functions from F that are identical outside $[k, k+1]$. By Φ_k we denote the set of all these functions. It is obvious that they form a group isomorphic to the group of all piecewise linear functions on $[0, 1]$ (with the properties mentioned in Section 1), that is, it is isomorphic to F. It is also easy to see that elements in different subgroups Φ_k commute with each other. Therefore, the groups Φ_k $(k \ge 1)$ generate a direct power of the group F. Conjugation by the element x_0, represented by the function given by $tx_0 = 2t$ $(t \in [0, 1])$, $tx_0 = t + 1$ $(t \ge 1)$, takes Φ_k to Φ_{k+1}. It is now clear that t and Φ_k $(k \ge 1)$ generate the restricted wreath product F wr $\mathbf{Z}$ in F.

It is not hard to find generators of the subgroup F wr $\mathbf{Z}$ of F in a diagram form and also in a normal form. In particular, the subgroup in F generated by elements x_0, $x_1x_2x_1^{-2}$, $x_1^2x_2x_1^{-3}$ will be isomorphic to F wr $\mathbf{Z}$. The reader can easily draw the diagrams representing these elements.

Let us define a sequence of groups by induction: $H_1 = \mathbf{Z}$, $H_{n+1} = H_n$ wr $\mathbf{Z}$. Thus groups $H_n = (\cdots(\mathbf{Z} \text{ wr } \mathbf{Z}) \text{ wr } \cdots) \text{ wr } \mathbf{Z}$, where $\mathbf{Z}$ occurs n times, are diagram groups by Theorem 11. The group H_n is soluble of degree n. Using Lemma 19 and elementary properties of wreath products, we have the following result that can be proved by induction on n. This statement was obtained by M. Brin (private communication), see also [2].

Corollary 20 *For any n, the group $H_n = (\cdots(\mathbf{Z} \text{ wr } \mathbf{Z}) \text{ wr } \cdots) \text{ wr } \mathbf{Z}$ is a soluble subgroup of degree n in R. Thompson's group F.*

Talking about wreath products, we would like to mention a fact about subgroups of R. Thompson's group F. It was shown in [3] that any subgroup of F is either metabelian or contains an infinite direct power of the group $\mathbf{Z}$. (In fact, one can replace the word "metabelian" by "abelian", see [6].) The proof given in [3, 6] uses representations of F by piecewise linear functions. Actually, the result is obtained for subgroups of some group which is bigger than F. It turns out that one can extract a stronger fact from this proof. Consider all piecewise linear continuous transformations of the unit interval $I = [0, 1]$ onto itself. We consider only mappings that preserve orientation and have finitely many breaks of the derivative. All these functions form a group with respect to composition. Let us denote this group by $PL_0(I)$. It contains F as a subgroup. We have the following alternative for subgroups of the group $PL_0(I)$.

Theorem 21 *Any subgroup of $PL_0(I)$ is either abelian, or contains an isomorphic copy of* $\mathbf{Z}$ wr $\mathbf{Z}$.

Proof. Our proof basicaly follows the proof or a weaker alternative from [3, 6]. For $f \in PL_0(I)$ by $\operatorname{supp} f$ we denote the set of all $t \in I$, for which $tf \neq t$. Let G be a non-abelian subgroup of $PL_0(I)$. Consider functions $f, g \in G$ such that $fg \neq gf$. Let $J = \operatorname{supp} f \cup \operatorname{supp} g$. It is obvious that J is a union of finitely many disjoint intervals $J_k = (a_k, b_k)$, $1 \le k \le m$. By definition, $[f, g] \neq 1$ in $PL_0(I)$. Then on some of intervals $J_1, \ldots, J_m$ our function $[f, g]$ is not the identity. Denote by $\nu(f, g)$ the number of such intervals. Without loss of generality, we can assume that the elements $f, g \in G$ which do not commute are chosen in such a way that the value $\nu(f, g)$ is the smallest possible. Let H be a subgroup of $PL_0(I)$ generated by f and g. By definition, the endpoints of J_1, $\ldots$, J_m are stable under f and g so each of these intervals is H-invariant.

An easy argument shows that for any $x, y \in J_k$ $(1 \le k \le m)$, where $x < y$, there exists a function $w \in H$ such that $xw > y$. Let us take the greatest upper bound z of the set $\{\, xh \mid h \in H \,\}$. It is clear that $a_i < z \le b_i$. If $z \neq b_i$ then either $zf \neq z$ or $zg \neq z$ by definition of the set J. Without loss of generality let $zf \neq z$. This inequality also holds in a small neighbourhood of the point z. Therefore one of the numbers zf or zf^{-1} is greater than z, a contradiction. Thus $z = b_i$. This implies that acting by some element of H one can make the image of x as close to b_i as one wishes which is what we had to prove.

Let us take an interval (a_i, b_i) $(1 \le i \le m)$ such that $[f, g]$ is not identical on it. It is easy to ee that the function $[f, g]$ is identical in some neighbourhood of each of the points a_i, b_i. Thus $\operatorname{supp}[f, g]$ is nonempty and it is contained in $[c_0, d_0]$, where $a_i < c_0 < d_0 < b$. According to the above, there exists a function $w \in H$ such that $d_0 < c_0 w < b$. Let us denote $[f, g]$ by h_0. For any $n \ge 1$, let $c_n = c_0 w^n$, $d_n = d_0 w^n$, $h_n = h_0^{w^n}$. It is obvious that $c_0 < d_0 < c_1 < d_1 < \cdots$, and $\operatorname{supp} h_n \cap J_i \subseteq [c_n, d_n]$. Therefore, for any $i, j \ge 0$, the commutator $[h_i, h_j]$ is identical on J_i. Suppose that $[h_i, h_j] \neq 1$ for some i, j. Since all the intervals $J_1, \ldots, J_m$ are H-invariant, it is clear that all the functions h_1, h_2, $\ldots$ are identical on all the intervals J_k $(1 \le k \le m)$ where the function h_0 is identical. Therefore, we can replace f, g by h_i, h_j obtaining $\nu(h_i, h_j) < \nu(f, g)$. This contradicts the choice of f, g. This proves that $[h_i, h_j] = 1$ for any $i, j \ge 0$.

So far we very closely followed the proof given in [6]. The conclusion in [6], is that elements h_0, h_1, h_2, $\ldots$ form a basis of a free abelian group. To prove a stronger statement of our theorem, it remains now to add that $h_n^w = h_{n+1}$ for all $n \ge 0$, so the elements h_0, w generate $\mathbf{Z}$ wr $\mathbf{Z} \subseteq G$.

The Theorem is proved.

Using Theorem 16, we have the following alternative for subgroups of R. Thompson's group F.

Corollary 22 *Any subgroup of R. Thompson's group F is either free abelian or contains the restricted wreath product $\mathbf{Z}$ wr $\mathbf{Z}$.*

Note that in the group of all piecewise linear functions, not every abelian subgroup is free abelian.

We can extract one more corollary from Theorem 21.

Corollary 23 *A non-abelian group with one defining relation cannot be a subgroup of the group $PL_0(I)$ (in particular, it cannot be a subgroup of R. Thompson's group F).*

Proof. In [7], A. A. Chebotar described all subgroups of one-relator groups that do not contain free subgroups of rank 2. They are: a) abelian subgroups, b) free product $\mathbf{Z}_2 * \mathbf{Z}_2$ of the cyclic group of order 2 by itself, and c) Baumslag – Solitar groups $B_{1k} = \langle\, a, b \mid b^{-1}ab = a^k \,\rangle$. It is clear that the group $\mathbf{Z}$ wr $\mathbf{Z}$ does not occur in this list. Thus a non-abelian one-relator group cannot be a subgroup of $PL_0(I)$ by Theorem 21.

The Corollary is proved.

Now let us consider the following interesting question: under what condition on a semigroup presentation $\mathcal{P}$ a diagram group over this presentation contains zz wr $\mathbf{Z}$ as a subgroup? The answer is given in the following theorem.

Theorem 24 *Let $\mathcal{P} = \langle\, \Sigma \mid \mathcal{R} \,\rangle$ be a semigroup presentation and let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. Then the following three conditions are equivalent.*

1. *The group $G = \mathcal{D}(\mathcal{P}, w)$ contains $\mathbf{Z}$ wr $\mathbf{Z}$ as a subgroup.*
2. *The group G contains elements a, b such that $[a, b] \neq 1$, $[a, a^b] = 1$ (in other words, there are two distinct elements in G that are conjugate and commute).*
3. *There are words x, y, z such that the equalities $xy = x$, $yz = z$, $xz = w$ hold modulo $\mathcal{P}$, and $\mathcal{D}(\mathcal{P}, y) \neq 1$.*

Proof. The proof uses the following scheme: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (2) is obvious and holds for any group G. The implication (3) $\Rightarrow$ (1) was proved in Theorem 18. It remains to show that (2) $\Rightarrow$ (3).

Suppose that $G = \mathcal{D}(\mathcal{P}, w)$ has elements a, b such that $[a, b] \neq 1$, $[a, a^b] = 1$. By Theorem 17, one can pass from the base w to some base v that equals w modulo $\mathcal{P}$ in such a way that the diagarams representing the two given commuting elements will be absolutely reduced and normal. Without loss of generality we can assume that a is represented by a diagram $A = A_1 + \cdots + A_m$ decomposed into the sum of components, and the element a^b is represented by a diagarm C that has a decomposition into the sum of the same number of components: $C = C_1 + \cdots + C_m$ by [12, Lemma 15.15]. By the same lemma, the element b which conjugates A and C is represented by a diagram B of the form $B_1 + \cdots + B_m$, where $C_i = B_i^{-1} A_i B_i$. Let $v = v_1 \dots v_m = v'_1 \dots v'_m$, where A_i, C_i are spherical diagrams with bases v_i, v'_i, respectively, and let B_i be a (v_i, v'_i)-diagram $(1 \le i \le m)$. It is obvious that $v_i = v'_i$ modulo $\mathcal{P}$ for all i.

We would like to prove that there exists an i from 1 to m such that the diagram A_i (and C_i as well) is nontrivial and the occurrences of the words v_i, v'_i in the word v do not overlap and do not contain each other. This would imply condition (3). Indeed, without loss of generality, let $v = pv_iqv'_ir$. Then the equalities $p = v_1 \dots v_{i-1} = v'_1 \dots v'_{i-1} = pv_iq$, $r = v'_{i+1} \dots v'_m = v_{i+1} \dots v_m = qv'_ir = qv_ir$ hold modulo $\mathcal{P}$. One can put $x = p$, $y = v_iq$, $z = v_ir$, and then the equalities $x = p = pv_iq = xy$, $z = v_ir = v_iqv_ir = yz$, $w = v = pv_iqv'_ir = pv_iqv_ir = pv_ir = xz$ will hold modulo $\mathcal{P}$ (that is, in the semigroup S). The diagram group over $\mathcal{P}$ with base $y = v_iq$ will be definitely nontrivial because there exists a nontrivial spherical diagram $A_i + \varepsilon(q)$ with this base.

Let us prove the existence of an i such that A_i is nontrivial, and the occurrences of the word v_i, v'_i in the word v have no common letters. Let us consider an arbitrary $1 \le i \le m$ such that A_i is nontrivial. Since C commutes with A, the diagram C can be presented as a sum $C' + D + C''$, where C', D, C'' are spherical diagrams with bases $v_1 \dots v_{i-1}$, v_i, $v_{i+1} \dots v_m$, respectively, and D commutes with A_i. Therefore, the diagram D consists of one component. The same is true for the diagram C_i. So, if the occurrences of v_i, v'_i have common letters, then diagrams D and C_i must coincide. This implies that A_i, C_i are powers of the same element Δ_i, and are conjugate. It follows from results of [12, Section 15] that $A_i = C_i$, and the occurences of v_i and v'_i coincide. It is now obvious that $A = C$. But this contradicts the assumption that $a \ne a^b$.

The proof is complete.

Let us make two remarks about the theorem we have just proved. Firstly, in the third condition we cannot avoid the condition that the diagram over $\mathcal{P}$ with base y is nontrivial. Without this condition, the diagram group may coincide with $\mathbf{Z}$. Note that the algorithm to verify whether the diagram group over a given finite presentation with given base is nontrivial, is unknown. Secondly, we have to note that if elements a, b of a diagram group are such that $[a, b] \ne 1$, $[a, a^b] = 1$, then the subgroup isomorphic to $\mathbf{Z}$ wr $\mathbf{Z}$, is not necessarily contained in the subgroup generated by a, b. An example illustrating that is given below in Section 5.

Finishing this Section, let us give a sufficient condition for a diagram group to contain R. Thompson's group F as a subgroup.

Theorem 25 *Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation, and let the semigroup S presented by $\mathcal{P}$ contain an idempotent. Then there is a word w such that the diagram group $G = \mathcal{D}(\mathcal{P}, w)$ contains R. Thompson's group F as a subgroup. Moreover, one can take any word w that represents an idempotent in S for such a base.*

Proof. The proof is quite easy. It is based on the fact that all proper homomorphic images of the group F are abelian (see [6]). Let us take a word w such that $ww = w$ modulo $\mathcal{P}$. Let us consider a reduced (w^2, w)-diagram Δ over $\mathcal{P}$. Now we construct a homomorphism from F to $\mathcal{D}(\mathcal{P}, w)$ in the following way. We use the fact that F is a diagram group over $\mathcal{Q} = \langle\, x \mid x^2 = x \,\rangle$. It is convenient to take the element x^5 as a base. To any diagram over $\mathcal{Q}$, we assign a diagram over $\mathcal{P}$, replacing the label x by w and filling in the cells of the form $x^2 = x$ by diagrams Δ. This rule defines a homomorphism from F to $\mathcal{D}(\mathcal{P}, w)$. Taking into account what we have said above, it is enough to check that the image of this homomorphism is not abelian. To show this, it suffices to compute the image of the commutator $[x_0, x_1]$. In the group $\mathcal{D}(\mathcal{Q}, x^5)$, this commutator is represented by the diagram $\varepsilon(x) + (x^2 = x) + (x = x^2) + \varepsilon(x)$. It is obvious that after we replace all (x^2, x)-cells in this diagram by copies of Δ, we get a diagram without dipoles. Thus the image of this commutator is not equal to the identity, so the image of F under the homomorphism is not abelian.

The Theorem is proved.

We do not know whether the condition on S to have an idempotent is also sufficient.

Problem 2 *Let $\mathcal{P} = \langle\, \Sigma \mid \mathcal{R} \,\rangle$ be a semigroup presentation and let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. Suppose that G contains R. Thompson's group F as a subgroup. Is it true that the semigroup S presented by $\mathcal{P}$ contains an idempotent?*

5 The Subgroup Conjecture

In this Section, we construct a counterexample to the Subgroup Conjecture, that is, we will construct a subgroup in a diagram group that is not a diagram group itself. Note that the first candidate to disprove the Subgroup Conjecture was the group F' — the commutator subgroup of R. Thompson's group F. However, it turned out that F' is a diagram group. This answered some open questions about diagram groups. Before proving the corresponding Theorem, let us make the following two remarks.

The first remark is about properties of semigroup diagrams over certain special presentations. Let we have a semigroup presentation of the form $\langle\, X \mid \mathcal{R} \,\rangle$, where all relations in $\mathcal{R}$ have the form $u = V$, where $u \in X$, $V \in X^+$. We assume that all left-hand sides of the relations are distinct and the right-hand sides contain more than one letter. It is known (see, for instance [10]) that any reduced diagram Δ over such a presentation can be uniquely decomposed into a concatenation: $\Delta = \Delta_1 \circ \Delta_2$, where Δ_1 corresponds to a derivation where only applications of relations of the form $u = V$ from $\mathcal{R}$ are used, and Δ_2 corresponds to a derivation where only applications of relations of the form $V = u$ are used, $(u = V) \in \mathcal{R}$. This fact can be easily proved by choosing the longest positive path from $\iota(\Delta)$ to $\tau(\Delta)$. It is easy to see that all cells "above" this path will correspond to relations $u = V$, and all cells "below" the path will correspond to $V = u$. We will call Δ_1 (resp. Δ_2) the *positive* (resp. *negative*) part of diagram Δ.

Note that the presentation $\langle\, x \mid x = x^2 \,\rangle$ satisfies the above conditions. The same holds for the presentation below from the statement of Theorem 26.

The second remark is about the structure of a commutator subgroup of a diagram group. It is described in [12, Theorem 11.3]. Let us recall this description. Let Δ be a

(w,w)-diagram over $\mathcal{P} = \langle\, \Sigma \mid \mathcal{R} \,\rangle$. By M, we denote the monoid presented by $\mathcal{P}$. We consider the free abelian group $\mathcal{A}$ with $M \times \mathcal{R} \times M$ as a free basis. For each vertex μ of diagram Δ we take any positive path from $\iota(\Delta)$ to μ. Its label defines an element in $\ell(\mu) \in M$. It is easy to show that this element does not depend on the choice of a path. Analogously, we define the element $r(\mu) \in M$ as the value of the label of any positive path from μ to $\tau(\Delta)$. Now to each cell π of the diagram Δ we assign an element $\delta \cdot (\ell(\iota(\pi)), u = v, r(\tau(\pi)))$, where $\delta = 1$, if $u = \varphi(\mathbf{top}(\pi))$, $v = \varphi(\mathbf{bot}(\pi))$, $(u,v) \in \mathcal{R}$ and $\delta = -1$, if $v = \varphi(\mathbf{top}(\pi))$, $u = \varphi(\mathbf{bot}(\pi))$, $(u,v) \in \mathcal{R}$. By $\rho(\Delta)$ we denote the sum of elements assigned to all the cells of diagram Δ. Thus ρ defines a homomorphism from the group $G = \mathcal{D}(\mathcal{P}, w)$ into $\mathcal{A}$. As it is shown in [12], the kernel of ρ is exactly G' — the commutator subgroup of G.

Theorem 26 *The commutator subgroup of R. Thompson's group F is a diagram group. Namely, $F' \cong \mathcal{D}(\mathcal{Q}, a_0b_0)$, where*

$$\mathcal{Q} = \langle\, x, a_i, b_i \ (i \geq 0) \mid x = xx, a_i = a_{i+1}x, b_i = xb_{i+1} \ (i \geq 0) \,\rangle.$$

Proof. Here is the direct proof of this proposition. Let us construct a mapping from $H = \mathcal{D}(\mathcal{Q}, a_0b_0)$ to $\mathcal{D}(\mathcal{P}, x^2)$, where $\mathcal{P} = \langle\, x \mid x = x^2 \,\rangle$. To each spherical diagram over $\mathcal{Q}$ with base a_0b_0 we assign a diagram that is obtained from the previous one replacing all its labels by x. It is clear that we get a spherical diagram over $\mathcal{P}$ with base x^2. Obviously, this induces a homomorphism $\psi\colon H \to F$ because $\mathcal{D}(\mathcal{P}, x^2) \cong F$. Our aim is to prove that the homomorphism ψ is injective and its image is F'.

Let Δ be a nontrivial reduced (a_0b_0, a_0b_0)-diagram over $\mathcal{Q}$. Its image under ψ cannot contain dipoles. Otherwise, the preimages of the cells that form a dipole in Δ, would form a dipole themselves. This implies that ψ is injective.

Let us check that $\psi(\Delta) \in F'$ for any reduced diagram Δ in H. The monoid M presented by $\langle\, x \mid x = xx \,\rangle$ consists of two elements 1 and x. A cell π of diagram $\Delta' = \psi(\Delta)$ satisfies $\ell(\iota(\pi)) = 1$ if and only if $\iota(\pi) = \iota(\Delta')$. Analogously, $r(\tau(\pi)) = 1$ if and only if $\tau(\pi) = \tau(\Delta')$. As we know, the diagram Δ' is reduced. It can be decomposed into a product $\Delta'_1 \circ \Delta'_2$ of its positive and negative part according to the first remark before the statement of the Theorem. It is easy to see that there are no cells π of the diagram Δ' can satisfy both of the conditions $\iota(\Delta) = \iota(\pi)$, $\tau(\Delta) = \tau(\pi)$ simultaneously (recall that the base of Δ' is x^2).

Consider the diagram Δ and decompose it into a concatenation of positive and negative part: $\Delta = \Delta_1 \circ \Delta_2$ (this is possible because $\mathcal{Q}$ satisfies conditions of the first remark before the statement of the theorem). Let a_n $(n \geq 0)$ be the label of the first edge of the path that cuts Δ into a positive part and a negative part. Then it is easy to extract from the form of the defining relations that all labels of edges which start at $\iota(\Delta)$, if one reads them from the top to the bottom of the diagram, are a_0, a_1, ..., a_n, ..., a_1, a_0. From this, it follows that the number of cells π such that $\iota(\pi) = \iota(\Delta)$, is the same for Δ_1 and Δ_2. ¿From this fact, we immediately conclude that it is the same for Δ', if we compare the number of these cells in Δ'_1 and Δ'_2. All these cells from Δ'_1 map to $(1, x = x^2, x)$ under ρ and all cells from Δ'_2 map to $-(1, x = x^2, x)$ (we emphasize the fact that we are talking about the cells whose initial vertices coincide with the initial vertex of the diagram).

An analogous argument can be applied to the cells whose terminal vertices coincide with the terminal vertex the diagram. Here the list of labels of the edges that come into $\tau(\Delta)$, if one reads them from top to bottom, is b_0, b_1, $\dots$, b_m, $\dots$, b_1, b_0 for some $m \ge 0$. Now one can use the fact that the cells of Δ_1' which we deal with in this paragraph map to $(x, x = x^2, 1)$ and the cells of Δ_2' map to $-(x, x = x^2, 1)$. It remains to note that Δ_1' has the same number of cells as Δ_2' since Δ' is spherical and all relations have the same form. Therefore, the number of cells π of Δ_1' that satisfy $\iota(\pi) \ne \iota(\Delta')$ and $\tau(\pi) \ne \tau(\Delta')$ is the same as the number of corresponding cells in Δ_2'. However, the first ones map to $(x, x = x^2, x)$ and the second ones map to $-(x, x = x^2, x)$. Hence $\rho(\Delta') = 0$. This proves that $\psi(\Delta) \in F'$.

It remains to show that every element in F' belongs to the image of ψ. In order to do that, let us take a reduced spherical diagram Δ' with base x^2 over $\langle\, x \mid x = x^2 \,\rangle$. It follows from $\Delta' \in F'$ that $\rho(\Delta') = 0$. We consider separately the cells of three types: those that map into a) $\pm(1, x = x^2, x)$, b) $\pm(x, x = x^2, 1)$, c) $\pm(x, x = x^2, x)$ under ρ, respectively.

Each cell belongs to exactly one of the three types. So it is clear that the sum over all cells of each of the types equals zero. This means that in the decomposition $\Delta' = \Delta_1' \circ \Delta_2'$ into positive and negative part, the number of cells of each of the types in Δ_1' will be the same as the number of cells of the same type in Δ_2'. Let us have $n \ge 0$ cells of the first type in each of the parts. Let us rename labels of the edges that go out of $\iota(\Delta')$, replacing them by a_0, a_1, $\dots$, a_n, $\dots$, a_1, a_0, respectively, from top to bottom. Analogously, let us have $m \ge 0$ cells of the second type in each of the parts. We rename labels of the edges that come into $\tau(\Delta')$, replacing them in the same way by b_0, b_1, $\dots$, b_m, $\dots$, b_1, b_0, respectively. The diagram we get as a result will be denoted by Δ.

It remains to note that Δ will be a spherical diagram over $\mathcal{Q}$ with base $a_0 b_0$. Indeed, any cell that has the same initial vertex as the one of Δ, corresponds to a relation of the form $a_i = a_{i+1}x$ $(i \ge 0)$ or its inverse. If a cell π of the positive part is taken, then $\mathbf{top}(\pi) = e$, $\mathbf{bot}(\pi) = e_1 e_2$, where e, e_1, e_2 are edges of the diagram. By our construction, the label of e equals a_i for some $i \ge 0$. It follows from the way we renamed the labels that e_1 has label a_{i+1}. Note that the initial vertex of e_2 is not $\iota(\Delta)$ because e_1 cannot be a loop. Also the terminal vertex of e_2 is not $\tau(\Delta)$. Otherwise the edge e connects the initial and the terminal vertex of Δ but this is impossible. Therefore, the label of e_2 is x. The arguments for the negative part of the diagram are analogous. Of course, any cell that has the same terminal vertex as Δ, corresponds to a relation of the form $b_i = x b_{i+1}$ $(i \ge 0)$ or its inverse. It is clear that ψ takes Δ into Δ'. This completes the proof.

Corollary 27 *A diagram group can be simple. In particular, there exist nontrivial diagram groups that coincide with their commutator subgroups and so they do not admit an LOG-presentation.*

The group F' is simple (see [6]). We proved in 26 that F' is a diagram group. In [12, Section 17] we asked if a nontrivial diagram group may coincide with its commutator subgroup. We have given a positive answer. This is interesting if to compare this result with [12, Theorem 12.1]. It was proved there that if all diagram groups over a semigroup presentation coincide with their commutator subgroups, then all of them are trivial. As we see, certain diagram groups may coincide with their commutator subgroups. As a by-product, we gave an answer to Problem 17.1 of the same paper: is it true that any diagram group admits an LOG-presentation? Recall that an LOG-presentation is a group presentation such that all defining relations have form $a = b^c$, where a, b and c are generators. The groups that admit such a presentation are called LOG-groups (this concept was introduced in [1], where these groups were characterized in terms of labelled oriented graphs). In Russian papers, one can often meet an equivalent terminology "C-group". Some interesting characterization of these groups was recently obtained by Yu. V. Kuzmin [19, 20]. We have already shown that any diagram group over a complete presentation (see [12]) admits an LOG-presentation (cf R. Thompson's group F). We also proved that any diagram group is a retract of an LOG-group. Since any LOG-group has $\mathbf{Z}$ as its homomorphic image, it cannot coincide with its commutator subgroup. Thus we proved that a diagram group may not have an LOG-presentation.

To construct a counterexample to the Subgroup Conjecture, we strongly use results of Section 2. In particular, we need Theorem 13 and Example 12.

Theorem 28 *There exist subgroups of digaram groups that are not diagram groups themselves. For instance, the following one-relator group*

$$\langle\, x, y \mid xy^2x = yx^2y \,\rangle$$

can be isomorphically represented by diagrams over a semigroup presentation but it is not a diagram group itself.

Proof. Let $L = \langle\, x, y \mid xy^2x = yx^2y \,\rangle$. We shall prove that L is not a diagram group. Consider the group L_∞ given by

$$L_\infty = \langle\, z_i \ (i \in \mathbf{Z}) \mid [z_i, z_{i+1}] = 1 \ (i \in \mathbf{Z}) \,\rangle.$$

The mapping ψ that takes z_i to z_{i+1} for all $i \in \mathbf{Z}$, obviously induces an automorphism of the group L_∞. Consider HNN-extension of the group L_∞ with stable letter t via automorphism ψ (this will be also a semidirect product of L_∞ and $\mathbf{Z}$). We obtain the group $\langle\, L_\infty, t \mid t^{-1}z_it = z_{i+1} \ (i \in \mathbf{Z}) \,\rangle$ that can be simplified to one-relator group $\langle\, t, z_0 \mid [z_0, z_0^t] = 1 \,\rangle$. Using Tietze transformations, one can transform it into L ($x = z_0t$, $y = t^{-1}$). So the group L_∞ is a subgroup of L.

Note that we could consider the group

$$L_0 = \langle\, z_i \ (i = 0, 1, 2, \ldots) \mid [z_i, z_{i+1}] = 1 \ (i = 0, 1, 2, \ldots) \,\rangle$$

instead of L_∞. The mapping ψ, where $\psi(z_i) = z_{i+1}$ ($i = 0, 1, 2, \ldots$), induces a monomorphism of the group L_0 into itself. Indeed, the mapping θ such that $\theta(z_0) = 1$, $\theta(z_i) = z_{i-1}$

$(i = 1, 2, \ldots)$ induces an endomorphism of the group L_0 and $\theta(\psi(z)) = z$ for any $z \in L_0$. If we take an HNN-extension of the group L_0 with stable letter t via monomorphism ψ, then we get the group $\langle\, L_0, t \mid t^{-1} z_i t = z_{i+1}\ (i = 0, 1, 2, \ldots)\,\rangle$, that can be transformed into L after simplifications. Remark that L_0 is obviously non-abelian (it maps onto a free group of rank 2 by the homomorphism which maps z_1 to 1, and z_i to 1 for $i = 3, 4, \ldots$). Hence L is also non-abelian, that is, $xy \ne yx$.

For $a = yx$, $b = x$ we have equality $[a, a^b] = 1$ in the group L and $[a, b] \ne 1$. If L is a diagram group then it satisfies Condition 2 of Theorem 24. Thus it also satisfies Condition 1, that is it contains $\mathbf{Z}$ wr $\mathbf{Z}$ as a subgroup. As we have mentioned above, in the proof of Corollary 23, the group $\mathbf{Z}$ wr $\mathbf{Z}$ cannot be a subgroup of a one-relator group because of the result of [7]. The contradiction we have obtained shows that L is not a diagram group.

It remains to show that L can be isomorphically embedded into a diagram group. We apply Theorem 13. It follows from it that the group $K = \mathcal{O}(\mathbf{Z}, \mathbf{Z})$ is a diagram group. It follows from the description given in Section 2 that K has a presentation in terms of generators g_i, h_i $(i \in \mathbf{Z})$, t and defining relations $[g_i, g_j] = [h_i, h_j] = 1$, $g_i^t = g_{i+1}$, $h_j^t = h_{j+1}$, where $i, j \in \mathbf{Z}$ and $[g_i, h_j] = 1$ for $i \le j$, $i, j \in \mathbf{Z}$. So it suffices to prove that the group $L = \langle\, x, y \mid [xy, yx] = 1\,\rangle$ is a subgroup in the diagram group $K = \mathcal{O}(\mathbf{Z}, \mathbf{Z})$. Consider the group

$$K_0 = \langle\, g_i, h_i\ (i \ge 0) \mid [g_i, g_j] = [h_i, h_j] = 1\ (i, j \ge 0),\ [g_i, h_j] = 1\ (j \ge i \ge 0)\,\rangle.$$

The map $g_i \mapsto g_{i+1}$, $h_j \mapsto h_{j+1}$ $(i, j \ge 0)$ can be extended to an endomorphism ψ of the group K_0. It is a monomorphism because the map $g_0, h_0 \mapsto 1$, $g_i \mapsto g_{i-1}$, $h_j \mapsto h_{j-1}$ $(i, j \ge 1)$ can be also extended to an endomorphism θ and $\theta(\psi(z)) = z$ for any $z \in K_0$. Therefore, one can consider an HNN-extension of the group K_0 with a stable letter t via monomorphism ψ. We obtain the group $\langle\, K_0, t \mid \psi(z) = z^t\ (z \in K_0)\,\rangle$ that has almost the same presentation as K with the only difference that the subscripts of the presentation of K run over all $\mathbf{Z}$. Adding new generators $g_i = g_0^{t^i}$, $h_j = h_0^{t^j}$ for negative i, j, we easily transform the presentation obtained above to the presentation of K. So it suffices to prove the following Lemma.

Lemma 29 *The subgroup in K generated by elements $g_i h_{i+1}$ $(i \ge 0)$ and t is isomorphic to L.*

Proof. The group K is an HNN-extension of the group K_0, that is, we add the letter t and relations $g_i^t = g_{i+1}$, $h_j^t = h_{j+1}$ $(i, j \ge 0)$ to its presentation. Define the sets $\mathcal{R}_k$ $(k \ge 0)$ of defining relations over the alphabet $\{\, z_0, h_0, z_1, h_1, \ldots\}$. For $\mathcal{R}_0$ we take the set of relations of the group L_0, that is, $\mathcal{R}_0 = \{\, [z_i, z_{i+1}] = 1\ (i \ge 0)\}$. Further, for $k \ge 1$ we put

$$\mathcal{R}_k = \{\, z_i^{h_k} = z_i\ (0 \le i < k)\,\} \cup \{\, z_i^{h_k} = z_i^{z_{k-1}}\ (i \ge k)\,\} \cup \{\, h_j^{h_k} = h_j\ (1 \le j < k)\,\}.$$

It is clear that $L_0 = \langle\, z_i\ (i \ge 0) \mid \mathcal{R}_0\,\rangle$. Let

$$L_k = \langle\, z_i\ (i \ge 0),\ h_j\ (1 \le j \le k) \mid \mathcal{R}_0 \cup \mathcal{R}_1 \ldots \cup \mathcal{R}_k\,\rangle$$

for $k \geq 1$. We shall prove that for any $k \geq 0$, the group L_{k+1} can be obtained from L_k by a suitable HNN-extension.

Consider the map ψ_k given by the following rules: $\psi_k(z_i) = z_i$ for $0 \leq i \leq k$, $\psi_k(z_i) = z_i^{z_k}$ for $i > k$, $\psi_k(h_j) = h_j$ for $1 \leq j \leq k$. Let us extend it to a homomorphism of the corresponding free group into the group L_k. Let us check that all relations of the group L_k will be equalities in L_k under ψ_k.

First of all we shall check that $\psi_k([z_i, z_{i+1}]) = 1$ for all $i \geq 0$. If $0 \leq i < k$, then $\psi_k([z_i, z_{i+1}]) = [z_i, z_{i+1}] = 1$ in L_k. The equality $\psi_k(z_i) = z_i^{z_k}$ holds also for $i = k$. Hence for all $i \geq k$ we also have $\psi_k([z_i, z_{i+1}]) = [z_i^{z_k}, z_{i+1}^{z_k}] = [z_i, z_{i+1}]^{z_k} = 1$. Now consider the other relations of L_k. They have one of the following three forms: $z_i^{h_j} = z_i$ for $0 \leq i < j \leq k$; $z_i^{h_j} = z_i^{z_{j-1}}$ for $i \geq j$, $1 \leq j \leq k$; $[h_i, h_j] = 1$ for $1 \leq i < j \leq k$. Considering three cases, we map each of the relations by ψ_k.

If $0 \leq i < j \leq k$ then we have $\psi_k(z_i^{h_j}) = \psi_k(z_i)^{\psi_k(h_j)} = z_i^{h_j} = z_i = \psi_k(z_i)$.

In the second case we will consider two subcases: $i \leq k$ and $i > k$. In the first subcase, that is, $1 \leq j \leq i \leq k$, we get $\psi_k(z_i^{h_j}) = \psi_k(z_i)^{\psi_k(h_j)} = z_i^{h_j} = z_i^{z_{j-1}} = \psi_k(z_i)^{\psi_k(z_{j-1})} = \psi_k(z_i^{z_{j-1}}$. In the second subcase, that is, $1 \leq j \leq k < i$, we have: $\psi_k(z_i^{h_j}) = \psi_k(z_i)^{\psi_k(h_j)} = (z_i^{z_k})^{h_j} = (z_i^{h_j})^{z_k^{h_j}} = (z_i^{z_{j-1}})^{z_k^{z_{j-1}}} = (z_i^{z_k})^{z_{j-1}} = \psi_k(z_i)^{\psi_k(z_{j-1})} = \psi_k(z_i^{z_{j-1}})$ (equality $z_k^{h_j} = z_k^{z_{j-1}}$ in the group L_k we used in these calculations, is a partial case of the relation of the second form for $i = k$).

In the third case everything is easy: $\psi_k([h_i, h_j]) = [\psi_k(h_i), \psi_k(h_j)] = [h_i, h_j] = 1$ for $1 \leq i < j \leq k$.

So ψ_k induces an endomorphism of the group L_k. Let us also introduce the map θ_k given by the rules $\theta_k(z_i) = z_i$ for $0 \leq i \leq k$, $\theta_k(z_i) = z_i^{z_k^{-1}}$ for $i > k$, $\psi_k(h_j) = h_j$ for $1 \leq j \leq k$. One can analogously check that θ_k induces an endomorphism of the group L_k. It is obvious that $\theta_k(\psi_k(z)) = \psi_k(\theta_k(z))$ for any $z \in L_k$. This means that ψ_k and θ_k are mutually inverse automorphisms of the group L_k.

Consider an HNN-extension of the group L_k with stable letter h_{k+1} via automorphism ψ_k of the group L_k. Its presentation is obtained from the one of L_k by adding h_{k+1} to the set of generators and adding relations of the form $\psi_k(z) = z^{h_{k+1}}$ to the set of defining relations, where z runs over all generators of L_k. These new relations form exactly the set $\mathcal{R}_{k+1}$. Therefore, this HNN-extension is the group L_{k+1}. In addition, we also have a natural embedding of L_k into L_{k+1} for $k \geq 0$.

We have a sequence of embedded subgroups

$$L_0 < L_1 < \cdots < L_k < L_{k+1} < \cdots ,$$

that give the group

$$\hat{L} = \langle\, z_i \ (i \geq 0),\ h_j \ (j \geq 1) \mid \mathcal{R}_0 \cup \mathcal{R}_1 \cup \cdots \mathcal{R}_k \cup \cdots \,\rangle$$

as a union of them. Let H be a subgroup generated by z_0 and all h_j $(j \geq 1)$. Adding h_0 as a stable letter, we construct an HNN-extension of the group $\hat{L}$ via identical endomorphism of H onto itself. That is, we add a new generator h_0 and relations $[z_0, h_0] = 1$, $[h_j, h_0] = 1$

for all $j \geq 1$. Let us describe explicitly the group $\bar{L}$ that we get as a result. It has generators z_i, h_i $(i \geq 0)$ subject to the following defining relations:

$$[z_i, z_{i+1}] = 1 \quad (i \geq 0),$$

$$[h_i, h_j] = 1 \quad (i, j \geq 0),$$

$$[z_i, h_j] = 1 \quad (0 \leq i < j),$$

$$[z_i, z_{j-1}h_j^{-1}] = 1 \quad (1 \leq j \leq i),$$

$$[z_0, h_0] = 1.$$

(we took the relations $\mathcal{R}_k$ for all $k \geq 0$ together with the relations added at the last step).

Let us introduce new generators $g_i = z_i h_{i+1}^{-1}$ $(i \geq 0)$. The elements g_i, h_i generate $\bar{L}$ so our aim is to describe relations of the group $\bar{L}$ in terms of these generators. Replacing elements z_i by $g_i h_{i+1}$ in the defining relations of the group $\bar{L}$, we get:

$$[g_i h_{i+1}, g_{i+1} h_{i+2}] = 1 \quad (i \geq 0), \tag{19}$$

$$[h_i, h_j] = 1 \quad (i, j \geq 0), \tag{20}$$

$$[g_i h_{i+1}, h_j] = 1 \quad (0 \leq i < j), \tag{21}$$

$$[g_i h_{i+1}, g_{j-1}] = 1 \quad (1 \leq j \leq i), \tag{22}$$

$$[g_0 h_1, h_0] = 1. \tag{23}$$

Since elements of the form h_i $(i \geq 0)$ pairwise commute, we can simplify (21), getting $[g_i, h_j] = 1$ for $0 \leq i < j$. Then in (22) the elements g_{j-1} h_{i+1} commute for $1 \leq j \leq i$ and so (22) reduces to $[g_i, g_{j-1}] = 1$ for all $1 \leq j \leq i$. This means that all elements g_i $(i \geq 0)$ pairwise commute. Let us simplify (19). Note that h_{i+2} commutes with the other three elements so it can be excluded. The equality $[g_i h_{i+1}, g_{i+1}] = 1$ we get is equivalent to $[h_{i+1}, g_{i+1}] = 1$ since g_i commutes with the other elements. Thus, simplifying (23), we obtain equalities $[g_i, h_i] = 1$ for all $i \geq 0$.

Let us summarize the above. The group $\bar{L}$ has generators g_i, h_i $(i \geq 0)$, where elements g_i pairwise commute. Elements of the form h_i also pairwise commute and g_i commutes with h_j whenever $i \leq j$. This means that the group $\bar{L}$ coincides with K_0. The group L_0, naturally embedded into $\bar{L}$, is generated by elements z_i $(i \geq 0)$. So the subgroup of $\bar{L}$ generated by $g_i h_{i+1} = z_i$ $(i \geq 0)$ is isomorphic to L_0.

The group K is an HNN-extension of the group K_0 via the monomorphism $\psi \colon K_0 \to K_0$ with t as a stable letter. Let us have a subgroup M_0 of K_0 such that $\psi(M_0) \subseteq M_0$. In this case, it is easy to show that the subgroup generated by M_0 and t will be the HNN-extension of M via the restriction of ψ on M_0.

Indeed, let us take such an HNN-extension. It has the form $M = \langle\, M_0, t \mid z^t = \psi(z)\ (z \in M_0)\,\rangle$. The map $t \mapsto t$, $z \to z$ for $z \in M_0$ induces a homomorphism ϕ from M to K. Since $zt = t\psi(z)$ we can represented any element of the group M in the form $t^\alpha z t^{-\beta}$, where $z \in M_0$, $\alpha, \beta \in \mathbf{Z}$, $\beta \geq 0$. Therefore, any element m in M is conjugated to an element of the form $t^\gamma z$ for some $\gamma \in \mathbf{Z}$, $z \in M_0$. If $m \neq 1$, then either $\gamma \neq 0$ or $z \neq 1$.

The element $t^\gamma z$ maps to an element in K of the same form under ψ. It follows from the elementary properties of HNN-extensions that it is not equal to 1 in K. Therefore, ϕ is an embedding of M into K and its image is exactly the subgroup of K generated by M_0 and t.

Note that the subgroup of K_0 generated by elements $z_i = g_i h_{i+1}$ $(i \geq 0)$ is invariant under ψ because $\psi(z_i) = z_{i+1}$ for all $i \geq 0$. So one can regard this subgroup (isomorphic to L_0) as M_0 and apply the arguments from the above paragraph. The corresponding HNN-extension of it is the subgroup of K generated by t and $g_i h_{i+1}$ $(i \geq 0)$. On the other hand, as we have mentioned in the beginning, this HNN-extension is exactly L.

The Lemma and Theorem 28 are proved.

There are not many known counterexamples to the Subgroup Conjecture. So it is natural to try to prove this conjecture under some restrictions on the subgroup. With respect to Theorem 26, we would like to ask a few questions.

Problem 3 *Is it true that any subgroup of R. Thompson's group F is a diagram group?*

Problem 4 *Is it true that the commutator subgroup of any diagram group is a diagram group?*

It is easy to see that the commutator subgroup of the group F satisfies the following condition. Let Δ be a diagram representing an element in F', and suppose that a conjugate diagram $\Psi^{-1}\Delta\Psi$ is a sum $\Gamma_1 + \Gamma_2$ of two nontrivial spherical diagrams with bases v_1, v_2, respectively. Then the diagrams $\Delta_1 = \Psi(\Gamma_1 + \varepsilon(v_2))\Psi^{-1}$ and $\Delta_2 = \Psi(\varepsilon(v_1) + \Gamma_2)\Psi^{-1}$ also belong to F'. It is easy to see that $\Delta = \Delta_1\Delta_2$, where Δ_1 and Δ_2 commute and do not belong to the same cyclic subgroup. Consider any subgroup H of a diagram group G that satisfies the above condition. We shall say that H is *closed* in G.

Problem 5 *Let H be a closed subgroup in a diagram group G. Is it true that H is a diagram group?*

If the answer to the next problem is positive, then this would imply that all word hyperbolic diagram groups are free.

Problem 6 *Let H be a subgroup in a diagram group G. Suppose that for any $h \in H$, $h \neq 1$, the centralizer $C_G(h)$ of h in G is cyclic. Does it imply that H is free (at least for the particular case $H = G$)?*

At the end of this Section let us consider an interesting family of groups. Let

$$G_n = \langle\, x_1, \ldots, x_n \mid [x_1, x_2] = [x_2, x_3] = \cdots = [x_{n-1}, x_n] = [x_n, x_1] = 1 \,\rangle.$$

It is easy to see that $G_1 = \mathbf{Z}$, $G_2 = \mathbf{Z} \times \mathbf{Z}$, $G_3 = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$, $G_4 = \mathcal{F}_2 \times \mathcal{F}_2$, where $\mathcal{F}_2$ is the free group of rank 2. All these groups can be obtained from $\mathbf{Z}$ using finite direct and free products. So these are diagram groups. However, the group G_5 is not a diagram group.

Theorem 30 *The groups G_n are not diagram groups for odd $n \geq 5$.*

Proof. Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation and let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. Consider any element $g \in G$ presented by a diagram Δ. Let us decompose Δ into the sum of spherical components. It was proved in [12] that the number of these components is an invariant of a diagram with respect to conjugation: see the remark after the proof of Lemma 15.15. One can see from the same Lemma that the number of nontrivial components is also an invariant. Thus one can introduce a function **comp**, denoted by $\mathbf{comp}\,(g)$, the number of nontrivial components of a diagram that represents an element $g \in G$.

Let us introduce a partial binary relation $\prec$ on G. Let $g_1, g_2 \in G$ be such that $\mathbf{comp}\,(g_1) = \mathbf{comp}\,(g_2) = 1$ (in particular, g_1, g_2 are nontrivial). We put $g_1 \prec g_2$ whenever diagrams Δ_1, Δ_2 that represent elements g_1, g_2 respectively, satisfy the following condition: there are words $x, y, z \in \Sigma^*$, $u, v \in \Sigma^+$, some $(w, xuyvz)$-diagaram Γ, simple absolutely reduced spherical diagrams Ψ_1 and Ψ_2 with bases u, v respectively such that $\Gamma^{-1}\Delta_1\Gamma = \varepsilon(x) + \Psi_1 + \varepsilon(yvz)$, $\Gamma^{-1}\Delta_2\Gamma = \varepsilon(xuy) + \Psi_2 + \varepsilon(z)$. It follows from this definition that if $g_1 \prec g_2$, then elements g_1, g_2 commute and generate a subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$ in G. In particular, they do not belong to the same cyclic subgroup. Thus the relation $\prec$ is antireflexive, that is, condition $g \prec g$ never holds for $g \in G$. Let us establish a few properties of $\prec$.

Lemma 31 *Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation and let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. The relation $\prec$ is transitive, that is, for any $g_1, g_2, g_3 \in G$ such that* $\mathbf{comp}\,(g_1) = \mathbf{comp}\,(g_2) = \mathbf{comp}\,(g_3) = 1$*, conditions $g_1 \prec g_2$ and $g_2 \prec g_3$ imply $g_1 \prec g_3$.*

Proof. Let element g_i in G be represented by a diagram Δ_i ($i = 1, 2, 3$). Since $g_1 \prec g_2$, there are words $x, y, z \in \Sigma^*$, $u, v \in \Sigma^+$, $(w, xuyvz)$-diagram Γ, simple absolutely reduced spherical diagrams Ψ_1 and Ψ_2 with bases u, v respectively, such that $\Gamma^{-1}\Delta_1\Gamma = \varepsilon(x) + \Psi_1 + \varepsilon(yvz)$, $\Gamma^{-1}\Delta_2\Gamma = \varepsilon(xuy) + \Psi_2 + \varepsilon(z)$. Since $g_2 \prec g_3$, there are words $x', y', z' \in \Sigma^*$, $u', v' \in \Sigma^+$, $(w, x'u'y'v'z')$-diagram Γ', simple absolutely reduced spherical diagrams Ψ_2' and Ψ_3' with bases u', v' respectively, such that $(\Gamma')^{-1}\Delta_2\Gamma' = \varepsilon(x') + \Psi_2' + \varepsilon(y'v'z')$, $(\Gamma')^{-1}\Delta_3\Gamma' = \varepsilon(x'u'y') + \Psi_3' + \varepsilon(z')$. It follows from these conditions that $\varepsilon(xuy) + \Psi_2 + \varepsilon(z) = \Theta^{-1}(\varepsilon(x') + \Psi_2' + \varepsilon(y'v'z'))\Theta$, where $\Theta = (\Gamma')^{-1}\Gamma$. Diagrams $\varepsilon(xuy) + \Psi_2 + \varepsilon(z)$ and $\varepsilon(x') + \Psi_2' + \varepsilon(y'v'z')$ are conjugate by an element Θ. It follows from [12, Lemma 15.15] that the components of these diagrams are conjugate respectively. In particular, words x' and z are nonempty. Applying this Lemma, we conclude that $\Theta = \Theta_1 + \Theta_2 + \Theta_3$, where Θ_1, Θ_2, Θ_3 are (x', xuy)-, (u', v)- and $(y'v'z', z)$-diagrams, respectively.

Let us now take the diagram $\Xi = \Gamma(\varepsilon(xuyv) + \Theta_3^{-1})$. It is clear that $\Xi = \Gamma'\Theta(\varepsilon(xuyv) + \Theta_3^{-1}) = \Gamma'(\Theta_1 + \Theta_2 + \Theta_3)(\varepsilon(xuyv) + \Theta_3^{-1}) = \Gamma'(\Theta_1 + \Theta_2 + \varepsilon(y'v'z'))$. We have $\Xi^{-1}\Delta_1\Xi = (\varepsilon(xuyv) + \Theta_3^{-1})^{-1}(\Gamma^{-1}\Delta_1\Gamma)(\varepsilon(xuyv) + \Theta_3^{-1}) = (\varepsilon(xuyv) + \Theta_3^{-1})^{-1}(\varepsilon(x) + \Psi_1 + \varepsilon(yvz))(\varepsilon(xuyv) + \Theta_3^{-1}) = \varepsilon(x) + \Psi_1 + \varepsilon(yvy'v'z')$, and $\Xi^{-1}\Delta_3\Xi = (\Theta_1 + \Theta_2 + \varepsilon(y'v'z'))^{-1}(\Gamma')^{-1}\Delta_3\Gamma'(\Theta_1 + \Theta_2 + \varepsilon(y'v'z')) = (\Theta_1^{-1} + \Theta_2^{-1} + \varepsilon(y'v'z'))(\varepsilon(x'u'y') + \Psi_3' + \varepsilon(z'))(\Theta_1 + \Theta_2 + \varepsilon(y'v'z')) =$

$\varepsilon(xuyvy') + \Psi_3' + \varepsilon(z')$. Thus conjugating diagrams Δ_1 and Δ_3 by a $(w, xuyvy'v'z')$-diagram Ξ, we represent them in the form enabling us to conclude that $g_1 \prec g_3$.

The proof is complete.

Lemma 31 implies that the relation $\prec$ is also antisymmetric, that is $g_1 \prec g_2$ excludes $g_2 \prec g_1$. Let us establish one more property of $\prec$.

Lemma 32 *Let $\mathcal{P} = \langle\, \Sigma \mid \mathcal{R}\,\rangle$ be a semigroup presentation and let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. We claim that for any commuting elements $g_1, g_2 \in G$ that do not belong to the same cyclic subgroup and satisfy $\mathbf{comp}\,(g_1) = \mathbf{comp}\,(g_2) = 1$, exactly one of the following conditions holds: $g_1 \prec g_2$ or $g_2 \prec g_1$.*

Proof. Let A_i be a diagram that represents an element $g_i \in G$ $(i = 1, 2)$. Since $[g_1, g_2] = 1$, we can apply Theorem 17 and find a word $v = v_1 \dots v_n$, spherical (v_j, v_j)-diagrams Δ_j $(1 \le j \le n)$, integers d_{ij} $(1 \le i \le 2,\ 1 \le j \le n)$ and some (w, v)-diagram Γ such that $\Gamma^{-1} A_i \Gamma = \Delta_1^{d_{i1}} + \dots + \Delta_n^{d_{in}}$, where diagrams Δ_j $(1 \le j \le n)$ are either trivial or simple absolutely reduced. The condition $\mathbf{comp}\,(g_1) = 1$ means that there is exactly one number j from 1 to n such that diagram $\Delta_j^{d_{1j}}$ is not trivial. Analogously, condition $\mathbf{comp}\,(g_2) = 1$ means that there exists exactly one number k from 1 to n such that diagram $\Delta_k^{d_{1k}}$ is not trivial. If $j = k$ then diagrams A_1, A_2 belong to the same cyclic subgroup of G but this is impossible. If $j < k$, then $g_1 \prec g_2$ by definition. If $k < j$ then $g_2 < g_1$.

The proof is complete.

Let us continue the proof of Theorem 30. Let $n = 2k + 1$, $k \ge 2$. Suppose that $G_n = \mathcal{D}(\mathcal{P}, w)$ is a diagram group over $\mathcal{P} = \langle\, \Sigma \mid \mathcal{R}\,\rangle$ with base w. First of all let us prove that $\mathbf{comp}\,(x_i) = 1$ for all generators x_i of G_n. Let us establish that the centralizer $C(x_i)$ of the element x_i $(1 \le i \le n)$ in G_n is the subgroup generated by elements x_{i-1}, x_i and x_{i+1}, isomorphic to the direct product $\mathcal{F}_2 \times \mathbf{Z}$ (subscripts are taken modulo n). By symmetry, it suffices to consider the centralizer of x_n.

Suppose that $C(x_n) \ne \mathrm{gp}\,\langle\, x_1, x_{n-1}, x_n\,\rangle$. Consider a group word W of minimal length in $x_1, \dots, x_n$ such that $W \in C(x_n)$, $W \notin \mathrm{gp}\,\langle\, x_1, x_{n-1}, x_n\,\rangle$. In particular, the word W is nonempty and it has neither nonempty initial nor nonempty terminal segment that belongs to $\mathrm{gp}\,\langle\, x_1, x_{n-1}, x_n\,\rangle$. The group G_n is an HNN-extension with base

$$\Gamma = \langle\, x_1, \dots, x_{n-1} \mid [x_1, x_2] = [x_2, x_3] = \dots = [x_{n-2}, x_{n-1}] = 1\,\rangle,$$

and stable letter x_n, with respect to the identical automorphism of the subgroup $\mathrm{gp}\,\langle\, x_1, x_{n-1}\,\rangle$ of Γ. Consider the element $x_n^{-1} W x_n W^{-1}$ of this HNN-extension. It equals 1 in the group G_n since $W \in C(x_n)$. By Britton's Lemma (see [22]), the word $x_n^{-1} W x_n W^{-1}$ has a subword of the form $U = x_n^{-\delta} V x_n^{\delta}$, where $\delta = \pm 1$, $V \in \mathrm{gp}\,\langle\, x_1, x_{n-1}\,\rangle$ is a word that does not contain $x_n^{\pm 1}$. Since the word W is chosen to have minimal length, U is not contained in $W^{\pm 1}$. Otherwise the occurrence of U can be replaced by an occurrence of the word V that is equal to U in G_n, decreasing the length of W. It is clear that V is nonempty since W cannot begin or end with $x_n^{\pm 1}$. Thus V is neither initial nor terminal segment of $W^{\pm 1}$. So it is clear that U does not occur in $x_n^{-1} W x_n W^{-1}$. We got a contradiction.

Thus $C(x_n) = \mathrm{gp}\,\langle\, x_1, x_{n-1}, x_n \,\rangle$. Consider a mapping of the alphabet $\{\, x_1, \ldots, x_n \,\}$ into G_n, sending each of the elements x_1, x_{n-1}, x_n to itself and senging all the other elements to 1. Extending this mapping to a homomorphism of the corresponding free group into G_n, we see that all defining relations of the group G_n are sent to 1. Thus we have an induced homomorphism $\phi\colon G_n \to G_n$. It is obvious that it is a retraction, that is, $\phi^2 = \phi$. On the one hand, the subgroup $\phi(G_n)$ of G_n equals $\mathrm{gp}\,\langle\, x_1, x_{n-1}, x_n \,\rangle$; on the other hand, this group is presented by relations of the group G_n with additional conditions $x_2 = \cdots = x_{n-2} = 1$. Thus for any $n > 3$ we have

$$\mathrm{gp}\,\langle\, x_1, x_{n-1}, x_n \,\rangle = \phi(G_n) = \langle\, x_1, x_{n-1}, x_n \mid [x_{n-1}, x_n] = [x_n, x_1] = 1 \,\rangle \cong \mathcal{F}_2 \times \mathbf{Z},$$

as desired.

So $C(x_i) \cong \mathcal{F}_2 \times \mathbf{Z}$ for all i from 1 to n; in particular, the centre of $C(x_i)$ is cyclic. If x_i were represented by a diagram with more than one nontrivial component, then its centralizer would have at least two direct summands isomorphic to $\mathbf{Z}$ by [12, Theorem 15.35]. So its centre would not be cyclic. Taking into account that x_i is nontrivial, we conclude that $\mathbf{comp}\,(x_i) = 1$. (Notice that we have not used yet that n is odd.)

Apply Lemma 32 and suppose without loss of generality that $x_1 \prec x_2$. Suppose that $x_2 \prec x_3$. Then Lemma 31 would imply that $x_1 \prec x_3$, so elements x_1 and x_3 commute. It is clear that these elements do not commute in G_n. The contradiction we obtain allows to apply Lemma 32 again and to conclude that $x_3 \prec x_2$. We will obtain a contradiction again if we suppose that $x_4 \prec x_3$. So in fact $x_3 \prec x_4$. Continuing in this way, we shall conclude that $x_{2k+1} \prec x_{2k}$, $x_{2k+1} \prec x_1$, $x_2 \prec x_1$. We have a contradiction.

The Theorem is proved.

It is reasonable to pose a question with respect to Theorem 30.

Problem 7 *For which n the groups G_n are diagram groups? For which n they are isomorphically representable by diagrams?*

If there is an odd $n \geq 5$ such that the group G_n is representable by diagrams, then we have one more counterexample to the Subgroup Conjecture. Otherwise we would have a generalization of Theorem 30.

6 Distortion of Subgroups in Diagram Groups

The problems that concern distortion in groups form a branch of geometric group theory under development (see [8, 24, 25]). Let us recall some definitions.

Let A be a group with finite set of generators X. In this case, for any $g \in G$ there exists an $n \geq 0$ and $x_1, \ldots, x_n \in X^{\pm 1}$ such that $g = x_1 \ldots x_n$. The least n with this property is called the *length* of the element g with respect to the generating set X and it is denoted by $|g|_X$.

If there are two functions ϕ, ψ from G to the set of all nonnegative integers, then we shall write $\phi \preceq \psi$, whenever there is a positive integer constant C such that $\phi(g) \leq C\psi(g)$

for all $g \in G$. If it holds $\phi \preceq \psi$ and $\psi \preceq \phi$ for the two functions simultaneously, then we call these functions *equivalent* and denote this fact by $\phi \sim \psi$. Obviously, $\sim$ is in fact the equivalence relation (one does not have to mix it with another equivalence relation that is often used when the Dehn functions are discussed). So, for the two functions one has $\phi \sim \psi$ if and only if there exisets a positive integer constant C such that

$$\frac{\phi(g)}{C} \leq \psi(g) \leq C\phi(g) \quad \text{for all } g \in G.$$

If X and Y are finite sets of generators of the same group A, then elementary arguments show that functions $|\ |_X$ and $|\ |_Y$ are equivalent.

Let we have two finitely generated groups A and B such that A is a subgroup of B. Let us fix some finite system of generators X for the group A and some finite system of generators Y for the group B. For any element $g \in A$ we define two numbers: $|g|_X$ and $|g|_Y$. Functions $|\ |_X$, $|\ |_Y$ can be regarded as functions on A. Later we will compare functions on two groups one embedded into another, with respect to $\preceq$, taking the corresponding restrictions of these functions. It follows from elementary reasons that $|\ |_Y \preceq |\ |_X$. If the converse is true, that is, $|\ |_X \preceq |\ |_Y$ holds, the we say that a subgroup A embeds into B *quasiisometrically* or *without distortion* (this happens, if $|\ |_X \sim |\ |_Y$). Note that the equivalence of two length functions $|\ |_X$ and $|\ |_Y$ does not depend on the choice of finite systems of generators X and Y. If to consider functions up to equivalence, then one can introduce length functions ℓ_A and ℓ_B in finitely generated groups A and B, respectively, that will depend of A and B only. The quasiisometricity of an embedding of A into B means that $\ell_A \sim \ell_B$.

Now consider a more general situation of an embedding of A into B for two finitely generated groups, $A \leq B$. Let we distinguish some finite generating sets X and Y in groups A and B respectively. One can consider the function

$$\operatorname{disto}(n) = \max_{|g|_Y \leq n} |g|_X,$$

that describes distortion of the subgroup A embedded into B. It is called the *distortion function* of the subgroup A in B. It is easy to find out that if we change the generating sets, then the distortion function $\operatorname{disto}(n)$ is not essentially changed. The reader can easily write down the corresponding inequalities. Thus we can talk about linear, quadratic, polynomial, exponential etc distortion. The question about distortion is aslo interesting with respect to the so called membership problem. Let we have two finitely generated groups A and B, where $A \leq B$. The *membership problem* of elements of the group B into the subgroup A is the question on the existence of an algorithm that decides, given a word on the generators of B, whether the element of B presented by this word belongs to A. The membership problem of elements of B into a subgroup A is decidable if and only if the distortion function $\operatorname{disto}(n)$ defined above is recursive (equivalently, has a recursive upper bound).

It is interesting to find the conditions under which all finitely generated subgroups of a given group will embed into it (or any its finitely generated subgroup) without distortion.

Free groups and abelian groups have this property. Even for the case of nilpotent groups the situation is quite different: in any nilpotent (non-abelian) torsion-free group there are cyclic subgroups that have distortion in them. (Note that diagram groups may have distorted subgroups in general: due to the classical result of Mikhailova [23], the group $\mathcal{F}_2 \times \mathcal{F}_2$ has finitely generated subgroups with undecidable membership problem, so they are distorted.)

Let us mention two recent results of Burillo [5]: he proved that every cyclic subgroup of R. Thompson's group F is embedded into it without distortion. Also he gave examples of quasi-isometric embeddings of groups $F \times \mathbf{Z}^n$ $(n \geq 1)$ and $F \times F$ into F.

It is thus natural to ask whether every finitely generated subgroup embeds quasi-isometrically into F. We give a negative answer. Namely, for any integer $d \geq 2$ we construct a finitely generated subgroup of F with distortion at least n^d. The fact that any cyclic subgroup of **every** finitely generated group representable by diagrams (including the case of F) embeds quasi-isometrically into it, follows easily from [12, Lemma 15.29]. We shall prove a more general result.

Theorem 33 *Let B be a finitely generated subgroup of a diagram group G and let A be a finitely generated abelian subgroup of B. Then A embeds into B quasi-isometrically.*

Proof. Note that diagram groups are torsion-free [12, Theorem 15.11] and so A is isomorphic to $\mathbf{Z}^m$ for some integer m. (This also follows from Theorem 15.) Let $G = \mathcal{D}(\mathcal{P}, w)$, where $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is a semigroup presentation. By $A_1, \ldots, A_m$ we denote spherical diagrams with base w presenting free generators of $A \cong \mathbf{Z}^m$. Since these elements are pairwise commutative in G, we can apply Theorem 17 to them. Thus we have a word $v = v_1 \ldots v_n$, sperical (v_j, v_j)-diagrams Δ_j $(1 \leq j \leq n)$, integers d_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ and some (w, v)-diagram Γ such that

$$\Gamma^{-1} A_i \Gamma = \Delta_1^{d_{i1}} + \cdots + \Delta_n^{d_{in}}$$

for all $1 \leq i \leq m$. Each of the diagrams $\Delta_1, \ldots, \Delta_n$ is either trivial or simple absolutely reduced. Conjugation by diagram Γ is an isomorphism of groups $G = \mathcal{D}(\mathcal{P}, w)$ and $\mathcal{D}(\mathcal{P}, v)$. Since the property of a subgroup to be embeddable quasi-isometrically is an invariant under isomorphism, we can assume without loss of generality that B is a subgroup of $\mathcal{D}(\mathcal{P}, v)$.

It suffices to prove that $\ell_A \preceq \ell_B$. Consider the diagrams

$$\Delta_j' = \varepsilon(v_1 \ldots v_{j-1}) + \Delta_j + \varepsilon(v_{j+1} \ldots v_n)$$

for all j from 1 to n. By $\Psi_1, \ldots, \Psi_r$ we denote those of diagrams $\Delta_1', \ldots, \Delta_n'$ that are nontrivial. The form a basis X of a free abelian group C. Since A embeds into C quasi-isometrically, we have $\ell_A \sim \ell_C$. For any element in the group $\mathcal{D}(\mathcal{P}, v)$ presented by a reduced diagram Δ, we denote by $\#(\Delta)$ the number of cells in Δ. Thus $\#$ is a function on the diagram group. Let Y be a finite generating set of the group B and let K be the greatest number of cells for diagrams in Y. Then it is obvious that $\#(\Delta) \leq K|\Delta|_Y$ for any diagram Δ in B. So we have $\# \preceq \ell_B$. Let $s_1, \ldots, s_r$ be arbitrary integers.

Consider the element $\Delta = \Psi_1^{s_1} \dots \Psi_r^{s_r}$ in C. All diagrams Δ_j $(1 \le j \le n)$ are absolutely reduced. So it follows easily from the definition of diagrams Ψ_k $(1 \le k \le r)$ that $\#(\Delta) = |s_1|\#(\Psi_1) + \dots + |s_r|\#(\Psi_r)$. Since $|\Delta|_X = |s_1| + \dots + |s_r|$, we can deduce inequality $|\Delta|_X \le \#(\Delta) \le K'|\Delta|_X$, where K' is the greatest number of cells for the diagrams in X. Therefore, $\ell_C \sim \#$.

Summarizing what we have said above, we conclude that $\ell_A \sim \ell_C \sim \# \preceq \ell_B$. Now obvious inequality $\ell_B \preceq \ell_A$ gives us the equivalence $\ell_A \sim \ell_B$.

The Theorem is proved.

Now consider R. Thompson's group F, take any its element $g \in F$ and its centralizer $C_F(g)$ in F. In [12, Corollary 15.36] we gave the description of centralizers in F: they are finite direct products of groups that isomorphic to either F or $\mathbf{Z}$. In particular, all of them are finitely generated. Remark that if an element $g \in F$ is presented by a diagram Δ, then to find its centralizer, one needs to find an absolutely reduced diagram Δ' conjugated to Δ (this can be done effectively by [12, Lemma 15.14]) and then decompose Δ' into a sum of (spherical) components. To each trivail component we assign the group F and to each nontrivial one we assign $\mathbf{Z}$. Then we take direct product of these groups. It is easy to see that the groups we get in this way are exactly groups of the form $F^m \times \mathbf{Z}^n$, where $0 \le m \le n+1$.

The Theorem below generalizes Burillo's results from [5], where it is shown that F has quasi-isometrically embedded subgroups isomorphic to $F \times F$ (Proposition 9), and for every $n \ge 1$ there are quasi-isometrically embedded subgroups isomorphic to $F \times \mathbf{Z}^n$ (Corollary 6). (Although the group $F \times F$ cannot be a centralizer of an element in F, it is embeddable without distortion into $F^2 \times \mathbf{Z}$, which is a centralizer of some element in F. This implies the first of results quoted above.)

Theorem 34 *For any element g in R. Thompson's group F, the centralizer $C_F(g)$ of this element embeds into F quasi-isometrically.*

First of all we need a lemma that can be deduced easily as a consequence of [5, Proposition 2]. But we give a direct proof of the fact we need.

Let $\mathcal{P} = \langle\, x \mid x^2 = x \,\rangle$. For any $k \ge 1$, by $\#_k(g)$ we denote the number of cells in in the (reduced) spherical diagram with base x^k that presents the element $g \in F \cong \mathcal{D}(\mathcal{P}, x^k)$. The number $|g|$ denotes the length of $g \in F$ with respect to the set $\{x_0, x_1\}$ of generators.

Lemma 35 *For any $g \in F$, the following inequalities hold:*

$$\frac{|g|}{3} \le \#_3(g) \le 2|g|.$$

For any k the function $\#_k$ is equivalent to the length function $|\ |$.

Proof. Diagrams that correspond to the paths $(x^2, x \to x^2, 1)(1, x^2 \to x, x^2)$ and $(x, x \to x^2, x)(1, x^2 \to x, x^2)$ have two cells each. They present the elements x_0, x_1 of R. Thompson's group $F \cong \mathcal{D}(\mathcal{P}, x^3)$. If g is an element of length n, then it can be presented by a diagram with base x^3 that has at most $2n$ cells. The second inequality is thus proved.

Let us prove the first inequality. Let an element g is presented by a diagram Δ with base x^3. The longest positive path from $\iota(\Delta)$ to $\tau(\Delta)$ cuts this diagram into two parts: positive and negative one. The number of cells in each of the parts is the same, let it be equal to m. Then $\#_3(g) = 2m$. It is easy to see that the longest positive path has length $m+3$. Represent g as a normal form $g = g_1 g_2^{-1}$, where each of the elements g_1, g_2 is positive, that is, it is a product of positive exponents of generators. Since $|g| \le |g_1| + |g_2|$, it suffices to estimate the length of g_1. (The lenght of g_2 can be estimated analogously.) So let $g_1 = x_0^j x_{i_1} \dots x_{i_s}$, where $s \ge 0$ and $1 \le i_1 \le \dots \le i_s$ is the normal form of g_1. For any $i \ge 1$, replace x_i by $x_0^{1-i} x_1 x_0^{i-1}$, which is equal to it in F. Then we have that g_1 equals in F to the word

$$x_0^{j-i_1+1} x_1 x_0^{i_1-i_2} x_1 \dots x_0^{i_{s-1}-i_s} x_1 x_0^{i_s-1},$$

that has length

$$|j-i_1+1| + (i_2-i_1) + \dots + (i_s - i_{s-1}) + i_s - 1 + s = 2i_s - i_1 + |j-i_1+1| + s - 1.$$

(If $s = 0$, then the length is just j.)

according to the procedure described in 1 (Example refTGNF), we have inequality $s + j \le m$. If $s \ge 1$, then the element x_{i_s} corresponds to the edge $(x^t, x \to x^2, x^{i_s})$ in the Squier complex, where $t \ge 1$, so Δ has a positive path labelled by x^{t+2+i_s}. This implies $t + 2 + i_s \le m + 3$ hence $i_s \le m$.

Let us consider two cases.

) $j \ge i_1 - 1$ or $s = 0$. We have $|g_1| \le 2i_s + j + s - 2i_1 \le 3m - 2$ for $s \ge 1$. If $s = 0$, then $|g_1| = j \le m$.

) $s \ne 0$, $j < i_1 - 1$. In this case $|g_1| \le 2i_s + s - j - 2 \le 3m - 2$.

Summarizing, we conclude that $|g_1| \le 3m$ for all cases. Also $|g_2| \le 3m$. Therefore, $|g| \le 6m = 3\#_3(g)$, what we had to prove.

Now it remains to note that $|\#_k(g) - \#_3(g)| \le 2|k-3|$ since the diagram that presents g in $\mathcal{D}(\mathcal{P}, x^k)$ can be obtained from Δ conjugating it by a diagram of $|k-3|$ cells. From what follows that functions $\#_k$ and $\#_3$ are equivalent (one needs to use that $\#_k(g) = 0$ if and only if $g = 1$.)

The Lemma is proved.

Proof of Theorem 34. Let $g \in F$ be an arbitrary element. Let us present it by an (x,x)-diagram and reduce this diagram to absolutely reduced form by conjugation. We get some diagram Δ with base x^k. By Lemma 35, the length function in F is equivalent to $\#_k$. Let $\Delta = \Delta_1 + \dots + \Delta_m$ is a decomposition of Δ into the sum of components, where Δ_i is a spherical diagram with base z_i ($1 \le i \le m$). Any element in the centralizer of Δ is equal to a sum of (z_i, z_i)-diagrams, and the ith summand commutes with Δ_i ($1 \le i \le m$). For any i from 1 to m, let $G_i = F$, if Δ_i is trivial and $G_i = \mathbf{Z}$, if Δ_i is nontrivial. Thus $C_F(g) \cong G_1 \times \dots \times G_m$. Choose a system of generators in each of the groups G_i: if $G_i = F$, then the system consists of x_0, x_1, and for $G_i = \mathbf{Z}$ the system consists of one element. these systems of generators form a generating set Y of the centralizer of Δ. It is clear that any element h in the centralizer can be uniquely presented in the form $h_1 \dots h_m$, where $h_i \in G_i$ for $1 \le i \le m$, and $|h|_Y = |h_1|_1 + \dots + |h_m|_m$ (by $|\ |_i$ we denote the length in G_i with respect to the generating set we have chosen). The number of cells in Δ is equal to the sum of numbers of cells in diagrams Δ_i ($1 \le i \le m$). So it follows from the equivalence of the length function and the number of cells that the function $|\ |_Y$ is equivalent to the length function in $F \cong \mathcal{D}(\mathcal{P}, x^k)$. This means that the embedding of $C_F(g)$ into F is quasi-isometric. ($G_i = F$, then we apply Lemma 35. In the case $G_i = \mathbf{Z}$ the equivalence of the length function and the number of cells is obvious.)

The Theorem is proved.

Before going to the proof of the next result about distorted subgroups in F, let us consider the following construction that has its preimage in [23]. Let H be a group generated by a finite set X and let R be a finite subset in H. By N we denote the normal closure of the set R in H. Consider the subgroup K in $H \times H$ generated by all elements of the form (x, x), where $x \in X$, and also all elements of the form $(r, 1)$, where $r \in R$. it is easy to see that for any $g, h \in H$, the element (g, h) is in K if and only if the cosets of g and h by the subgroup N are equal. (This is proved in the same way as in [23]; see also [22].)

It is possible to consider an analog of the Dehn function in this situation. For any element $g \in N$ by $k(g)$ we denote the least k such that the element g is equal in H to a product of k elements conjugated in H to elements in R or their inverses. Let

$$\Phi(n) = \max_{|g| \le n} k(g),$$

where $|g|$ is the length of g with respect to the set X of generators. This function can be call a (relative) *Dehn function* of presentation $\langle X \mid R \rangle$ *with respect* to H; it is clear that if H is free, then we have standard Dehn function.

Let Y denote the above set of generators of K. Suppose that the element $(g, 1) \in K$ can be presented as a product of m elements from $Y^{\pm 1}$. Then we have an equality

$$(g, 1) = (u_0, u_0)(r_1, 1)^{\varepsilon_1}(u_1, u_1) \dots (r_m, 1)^{\varepsilon_m}(u_m, u_m),$$

that holds in K, where $u_0, u_1, \dots, u_m \in H$, $r_1, \dots, r_m \in R$, $\varepsilon_1, \dots, \varepsilon_m = \pm 1$. Therefore, equalities $g = u_0 r_1^{\varepsilon_1} u_1 \dots r_m^{\varepsilon_m} u_m$, $1 = u_0 u_1 \dots u_m$ hold in H. Then

$$g = u_0 u_1 \dots u_m r_1^{\varepsilon_1 u_1 \dots u_m} \dots r_m^{\varepsilon_m u_m} = r_1^{\varepsilon_1 u_1 \dots u_m} \dots r_m^{\varepsilon_m u_m}$$

. So the inequality $k(g) \le m$ holds. In particular, representing $(g, 1)$ as a product of the least number of generators in $Y^{\pm 1}$, we get the inequality $k(g) \le |(g, 1)|_K$. For each positive integer n we have an element $g \in H$ such that $|g|_X \le n$ and $\Phi(n) = k(g)$. The group $H \times H$ has the following natural set of generators: $Z = (X \times \{1\}) \cup (\{1\} \times X)$. It is clear that $|(g, 1)|_Z \le |g|_X \le n$, but we have $|(g, 1)|_K \ge k(g) = \Phi(n)$. It follows from the definition of the distortion function that $\mathrm{disto}\,(n) \ge \Phi(n)$, where we embed K into $H \times H$. We proved the following lemma.

Lemma 36 *Let H be a group generated by a finite set X, let R be a finite subset of H. Consider the subgroup K of $H \times H$ generated by the set Y that consists of all elements (x,x) ($x \in X$) and all elements of the form $(r,1)$ ($r \in R$). Then inequality* $\mathrm{disto}\,(n) \ge \Phi(n)$ *holds, where* $\mathrm{disto}\,(n)$ *is the distortion function for the embedding of the group K generated by Y into the group $H \times H$ generated by $Z = (X \times \{1\}) \cup (\{1\} \times X)$. Here $\Phi(n)$ is the relative Dehn function of presentation $\langle\, X \mid R \,\rangle$ with respect to H.*

An important property of R. Thompson's group F is that $F \times F$ can be embedded into F. So in order to obtain distorted subgroups in F we need to take such a subgroup H with finite generating set X and a finite subset R of H such that the Dehn function of $\langle\, X \mid R \,\rangle$ with respect to H will be overlinear. Then, in the above notation, we shall get an embedding of K into the group $H \times H$, which is in turn embeddable into $F \times F$ (and so it embeds into F). The embedding of K into F will not be quasi-isometric. Note that if to take $H = F$, $R = \{\, [x_0, x_1] \,\}$, then the **relative** Dehn function will be linear though the standard Dehn function (with respect to the free group on $\{x_0, x_1\}$) will be quadratic. Now we will give an example how to construct a subgroup in F with at least quadratic distortion. The last Theorem in this Section will be a generalization of this example.

Example 37 Let $H = \mathbf{Z}$ wr $\mathbf{Z}$ be a subgroup of F constructed in Section 4. Denote its generators by a and b. Let the elements $a_n = a^{b^n}$ ($n \in \mathbf{Z}$) form a basis of the free abelian subgroup. For R we take the set of a single element $[a,b] = a_0^{-1}a_1$. Conjugating this element by all elements in H, we shall get all elements of the form $c_i = a_i^{-1}a_{i+1}$ ($i \in \mathbf{Z}$). It is obvious that all elements of the form c_i are also a basis of the free abelian group. Let $g_n = [a^n, b^n] = a_0^{-n}a_n^n \in H$. The length of g_n with respect to $\{a,b\}$ does not exceed $4n$. At the same time, we have equality $g_n = c_0^n c_1^n \ldots c_{n-1}^n$, which shows that g_n can be presented as a product of n^2 elements of the form c_i ($i \in \mathbf{Z}$). Since the elements c_i form a basis of a free abelian subgroup, it follows that g_n cannot be presented as a product of less than n^2 elements of the form $c_i^{\pm 1}$. Therefore, the Dehn function $\Phi(n)$ of presentation $\langle\, a, b \mid [a,b] \,\rangle$ with respect to H satisfies inequality $\Phi(4n) \ge n^2$. Let K be a subgroup of $H \times H$ generated by (a,a), (b,b), $([a,b],1)$. Lemma 36 shows that the distortion function disto that characterizes the embedding of K into $H \times H$, is at least quadratic. In particular, K embeds into $H \times H$ with distortion (that is, the embedding is not quasi-isometric). It remains to embed $H \times H$ into $F \times F$ and then into F. Taking into account that $\ell_F \preceq \ell_{H \times H}$, we obtain that K embeds into F with distortion.

One can give explicit expressions (in terms of normal forms) of the generators of K as a subgroup in F. The elements $a = x_1x_2x_1^{-2}$ and $b = x_0$ generate in F a subgroup isomorphic to $\mathbf{Z}$ wr $\mathbf{Z}$. The rules $(x_0,1) \mapsto x_1x_2x_1^{-2}$, $(x_1,1) \mapsto x_1^2x_2x_1^{-3}$, $(1,x_0) \mapsto x_2x_3x_2^{-2}$, $(1,x_1) \mapsto x_2^2x_3x_2^{-3}$ give an embedding of $F \times F$ into F. Using that, it is easy to compute the generators of K. The following elements of F generate the subgroup isomorphic to K:

$$x_1^2x_2^2x_6^2x_7^2x_8^{-1}x_7^{-1}x_6^{-2}x_3^{-1}x_2^{-1}x_1^{-2},$$

$$x_1x_2x_4x_5x_4^{-2}x_1^{-2},$$

$$x_1^3 x_2^2 x_5 x_6 x_5^{-2} x_3^{-1} x_2^{-1} x_1^{-3}.$$

Now let us prove the result about distorted subgroups of F in its general form.

Theorem 38 *For any $d \geq 2$, there exists a finitely generated subgroup K_d of R. Thompson's group F such that the corresponding distortion function satisfies inequality $n^d \preceq$ disto(n).*

Proof. Define the groups H_k ($k \geq 0$) by induction on k in the following way. Let $H_0 = 1$, $H_{k+1} = H_k$ wr $\langle a_{k+1} \rangle$ for $k \geq 0$, where all groups $\langle a_k \rangle$ are infinite cyclic. According to Corollary 20, all of them are embeddable into F. Let us fix an integer $d \geq 2$ and consider the group $H_d \times H_d$, which is also embeddable into F. For any integers k, n define an element $g_k(n)$ as a left-normalized commutator

$$g_k(n) = [a_1^n, a_2^n, \ldots, a_k^n],$$

defined by induction on k: $g_1(n) = a_1^n$, $g_{k+1}(n) = [g_k(n), a_{k+1}^n]$ for $k \geq 1$.

The element $g_k(1)$ will be denoted by g_k. For R_d we take the set of a single element $g_d = [a_1, a_2, \ldots, a_d]$.

The elements in H_d of the form

$$a_i(t_1, \ldots, t_r) = a_i^{a_{i+1}^{t_1} \ldots a_{i+r}^{t_r}},$$

where $1 \leq i \leq d$, $0 \leq r \leq d-i$, $t_1, \ldots, t_r \in \mathbf{Z}$, will be called *basic*. If $r = 0$ then we just have a_i. Obviously, $a_i = a_i(0) = a_i(0,0) = \cdots$ and so on. In general, we can add zeroes on the right to the sequence $t_1, \ldots, t_r$ in such a way that the total number of arguments in brackets after a_i do not exceed $d-i$.

Consider two elements $w_i = a_i(s_{i+1}, \ldots, s_d)$ and $w_j = a_j(t_{j+1}, \ldots, t_d)$, where $1 \leq i \leq j \leq d$, and s_k ($i < k \leq d$), t_k ($j < k \leq d$) are integers. (It is easy to see that each pair of basic elements can be presented in this form.) It is clear that if the sequence t_{j+1}, $\ldots$, t_d is not an end of the sequence s_{i+1}, $\ldots$, s_d, then the elements w_i and w_j commute. Indeed, in this case one can choose the biggest k such that $s_k \neq t_k$. Conjugation by the inverse element to $a_{k+1}^{s_{k+1}} \ldots a_d^{s_d} = a_{k+1}^{t_{k+1}} \ldots a_d^{t_d}$ takes elements w_i, w_j into the elements $w_i' = a_i(s_{i+1} \ldots s_k) \in H_{k-1}^{a_k^{s_k}}$, $w_j' = a_j(t_{j+1} \ldots t_k) \in H_{k-1}^{a_k^{t_k}}$, respectively. But it is clear from the elementary properties of wreath products that the subgroups G^{z^s} and G^{z^t} of G wr $\langle z \rangle$, where z generates $\mathbf{Z}$, commute elementwise for any $s \neq t$. Now, if the sequence t_{j+1}, $\ldots$, t_d is the end of s_{i+1}, $\ldots$, s_d, that is, $s_k = t_k$ for $j < k \leq d$, then elements w_i and w_j coincide in the case $i = j$; in the case $i < j$ one can write them as $w_i = a_i(s_{i+1}, \ldots, s_j)^v$, $w_j = a_j^v$, where $v = a_{j+1}^{s_{j+1}} \ldots a_d^{s_d}$. Then for each $\ell \in \mathbf{Z}$ one has equalities

$$\begin{aligned} w_i^{w_j^\ell} &= (a_i(s_{i+1}, \ldots, s_j)^v)^{a_j^{\ell v}} = \left(a_i(s_{i+1}, \ldots, s_j)^{a_j^\ell}\right)^v = a_i^{a_{i+1}^{s_{i+1}} \ldots a_j^{s_j+\ell} v} \\ &= a_i(s_{i+1}, \ldots, s_j + \ell, s_{j+1}, \ldots, s_d). \end{aligned} \tag{24}$$

So we have a rule how to conjugate one basic element by another basic element.

Let $1 \le k \le d$. Consider the normal closure M_k of the element a_1 in H_k. It follows from the above that M_k is an abelian group freely generated by the set of elements

$$a_1(s_2, \ldots, s_k) = a_1^{a_2^{s_2} \ldots a_k^{s_k}},$$

where $s_2, \ldots, s_k \in \mathbf{Z}$. It is possible to define a homomorphism $\phi_k : M_k \to \mathbf{Z}$ from M_k into the **additive** group $\mathbf{Z}$ as follows: $\phi_k(a_1(s_2, \ldots, s_k)) = s_2 \ldots s_k$. From this definition we have that for any $k > 1$, $h \in M_{k-1}$ and for any $\ell \in \mathbf{Z}$ the following equality holds:

$$\phi_k(h^{a_k^\ell}) = \ell \phi_{k-1}(h).$$

In particular, ϕ_k equals zero on M_{k-1}.

Our aim is to establish the two facts.

1) For any $h \in H_k$ the equality $\phi_k(g_k^h) = 1$ holds.

Note that g_k obviously belongs to M_k so we can apply ϕ_k to any element conjugated to g_k.

2) If $1 \le k \le d$, then the element $g_k(n)$ belongs to the normal closure of the element g_k and $\phi_k(g_k(n)) = n^k$.

First we shall deduce the conclusion of our Theorem from these facts. The elements $a_1, \ldots, a_d$ generate the subgroup H_d. The length of $g_d(n)$ with respect to these generators does not exceed Dn, where $D = 3 \cdot 2^{d-1} - 2$ is a constant that does not depend on n. From the above two facts it is clear that the element $g_k(n)$, being a product of conjugates to g_d, cannot be presented as a product of less than n^d factors that are conjugates to g_d or their inverses. In the notation of Lemma 36, this gives inequality $\Phi(Dn) \ge n^d$. Applying this Lemma, we get $n^d \preceq \operatorname{disto}(n)$.

So let us prove the first of the above facts. We proceed by induction on k. If $k = 1$, then $g_1 = a_1 \in M_1$ and $g_1^h = a_1$ for any $h \in H_1 = \langle a_1 \rangle$. By definition, $\phi_1(a_1) = 1$. Let $k > 1$, $h \in H_k$. Then $g_k = [g_{k-1}, a_k] = g_{k-1}^{-1} g_{k-1}^{a_k} \in M_k$ since $g_{k-1} \in M_{k-1}$ by the inductive assumption. We have equalities

$$\phi_k(g_k^h) = \phi_k\left([g_{k-1}, a_k]^h\right) = \phi_k(g_{k-1}^{-h} g_{k-1}^{a_k h}) = \phi_k(g_{k-1}^{-h}) + \phi_k(g_{k-1}^{a_k h}).$$

Since $\phi_k = 0$ on M_{k-1}, the first summand equals zero. Further, the elements $g_{k-1}^{a_k} \in H_{k-1}^{a_k}$ and $h \in H_{k-1}$ commute, what follows from the definition of a wreath product. Therefore, $g_{k-1}^{a_k h} = g_{k-1}^{a_k}$. It follows from the above properties of ϕ_k that for any $g \in M_{k-1}$ we have $\phi_k(g^{a_k}) = \phi_{k-1}(g)$. So the second summand equals $\phi_k(g_{k-1}^{a_k}) = \phi_{k-1}(g_{k-1}) = 1$ because $g_{k-1} \in M_{k-1}$. As a result, $\phi_k(g_k^h) = 1$, what we had to prove.

Let us prove the second fact. By N_k we denote the normal closure of g_k in H_k. Let us prove by induction on k that $g_k(n) \in N_k$. This is obvious for $k = 1$ since $g_1(n) = a_1^n = g_1^n$. Let $k > 1$, and let the fact is true for all values of the parameter less than k. Since $g_k = [g_{k-1}, a_k]$, we have equality $g_{k-1}^{a_k} = g_{k-1}$ modulo N_k. In the group H_k, any element in $H_{k-1}^{a_k}$ commutes with any element in H_{k-1}. Therefore, g_{k-1} centralizes H_{k-1} modulo N_k. Then, modulo N_k, any element in the normal closure of g_{k-1} is some power of g_{k-1}.

In particular, this is true for the element $g_{k-1}(n)$ by the inductive assumption. Since g_{k-1} and a_k commute modulo N_k, we deduce that $g_k(n) = [g_{k-1}(n), a_k^n]$ equals 1 in the quotient group H_k/N_k, that is, $g_k(n) \in N_k$.

Now we prove that $\phi_k(g_k(n)) = n^k$ for $1 \le k \le d$ by induction on k. For $k = 1$ we get $\phi_1(g_1(n)) = \phi_1(a_1^n) = n\phi_1(a_1) = n$. Let $k > 1$; suppose that $\phi_{k-1}(g_{k-1}(n)) = n^{k-1}$. Then $\phi_k(g_k(n)) = \phi_k([g_{k-1}(n), a_k^n]) = \phi_k(g_{k-1}(n)^{-1}) + \phi_k(g_{k-1}(n)^{a_k^n}) = n\phi_{k-1}(g_{k-1}(n)) = n \cdot n^{k-1} = n^k$ (we have used the properties of ϕ_k, the fact that $g_{k-1}(n) \in M_{k-1}$ and the inductive assumption).

The Theorem is proved.

It is an interesting question what else functions may be distortion functions of finitely generated subgroups of F. In particular, it is very interesting if such a distortion function may not have a recursive upper bound. Let us give an equivalent form of this problem.

Problem 8 *Does R. Thompson's group F have a finitely generated subgroup with unsolvable membership problem?*

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