Polynomial maps over fields and residually finite groups

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Lecture 2. Some small cancelation theory and probability.

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- Residually finite,
- Virtually residually (finite p-)group for all but finitely many primes p,
- Coherent (that is all finitely generated subgroups are finitely presented).

Ken Brown's results

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- If $k = 2$ and one of the two support lines of w that is parallel to $\vec{OM}$ intersects w in a single vertex or a single edge, then G is an ascending HNN extension of a free group.
- If $k > 2$ then G is never an ascending HNN extension of a free group.

The Dunfield-Thurston result

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Hence $p_{good} < 1$.

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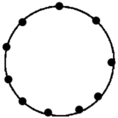
Proof



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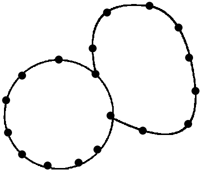

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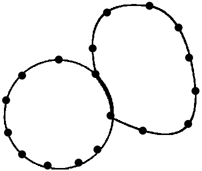

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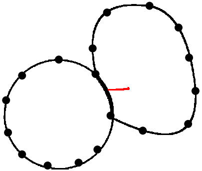

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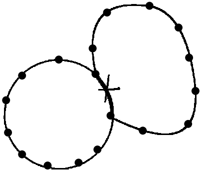

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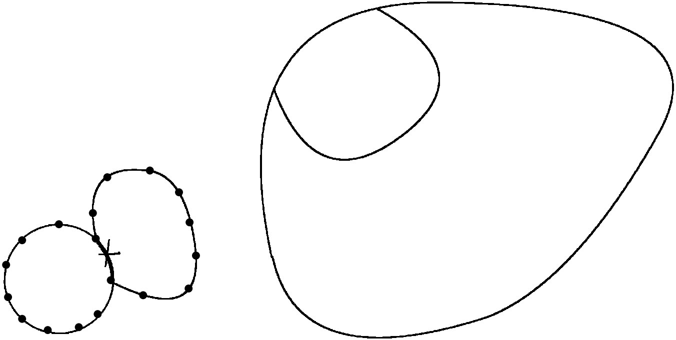

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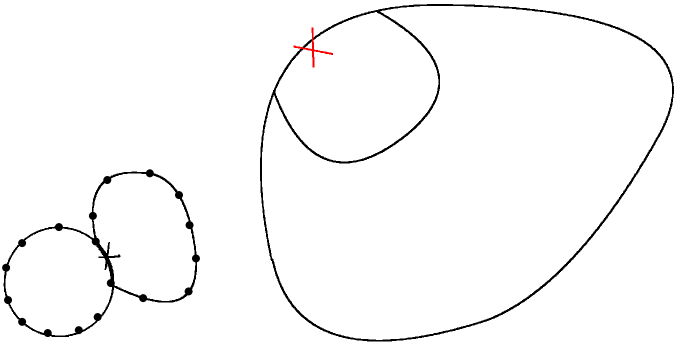

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$$\begin{aligned} w_1 &= aba^2b...a^nba^{n+1}ba^{-n-1}ba^{-n}b...a^{-2}ba^{-1}b \\ w_i &= ab^ia^2b^i...a^nb^ia^{-n}b^i...a^{-2}b^ia^{-1}b^i, \quad \text{for} \quad 1 < i < k \\ w_k &= ab^ka^2b^k...a^nb^ka^{-n}b^k...a^{-2}b^k \end{aligned}$$

Brownian Motion

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That is Brownian motion is a continuous Markov stationary process with normally distributed increments.

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We are using Rivin's Central Limit Theorem for cyclically reduced walks.

Convex hull of Brownian motion and maximal Magnus indices

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Convex hulls and maximal indices, continued

Step 1. We prove that the number of vertices of Δ is growing (a.s.) with the length of w (here it is used that $k \geq 3$).

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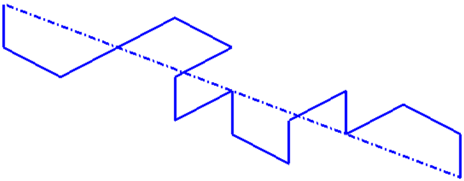
Step 2. For every vertex of Δ for any 'bad" walk w' or length r we construct (in a bijective manner) a "good" walk w' of length $r+4$. This implies that the number of vertices of "bad" walks is bounded if the probability of a "bad" walk is > 0.

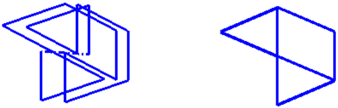
Illustration of Step 2

Here is the walk in $\mathbb{Z}^3$ corresponding to the word

$$cb^{-1}acac^{-1}b^{-1}caca^{-1}b^{-1}aab^{-1}c.$$

And its projection onto $\mathbb{R}^2$





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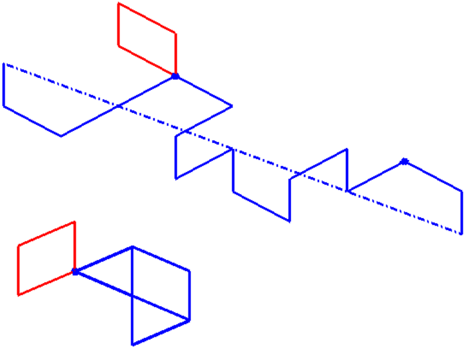

Homework

HW 1. We know that the group $\langle x, y, t \mid txt^{-1} = xy, tyt^{-1} = yx\rangle$ is hyperbolic (A. Minasyan). By Olshanskii, it must have infinitely many non-abelian finite simple homomorphic images. Find one. The group has the one-relation presentation $\langle x, t \mid [x, t, t] = x\rangle$.