Polynomial maps over fields and residually finite groups

Mark Sapir

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Iva Kozáková, Mark Sapir, Almost all one-relator groups with at least three generators are residually finite. preprint, arXiv math0809.4693, 2008.

Lecture 1. Around 1-related groups.

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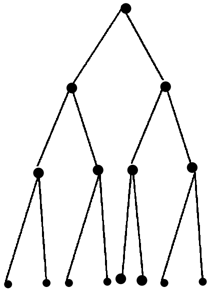
Examples. $\mathbb{Z}$, F_k, linear groups are residually finite. $\mathbb{Q}$, infinite simple groups, free Burnside groups of sufficiently large exponents are not residually finite. Groups acting faithfully on rooted locally finite trees are residually finite.

Rooted trees



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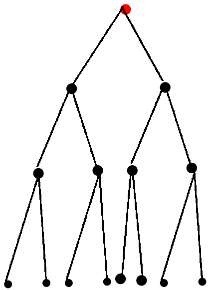

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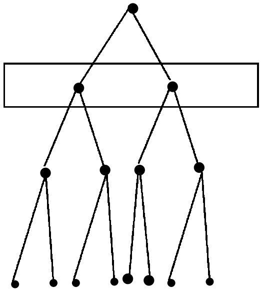

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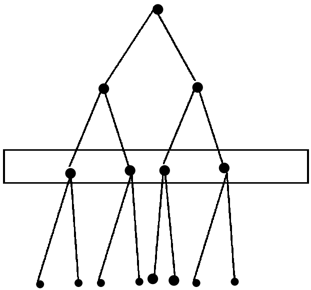

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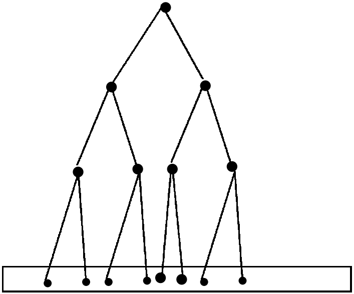

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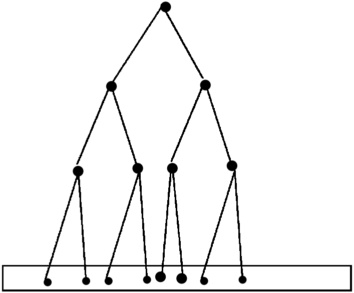

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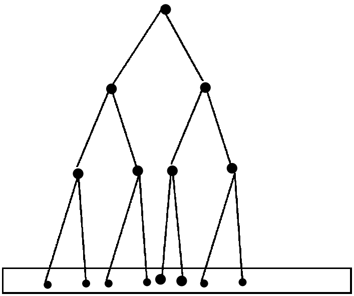

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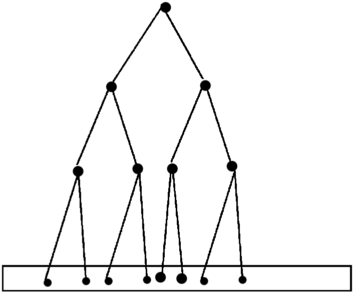

Conversely every finitely generated residually finite group acts faithfully on a locally finite rooted tree.

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These models are equivalent. $3 \equiv 1$: I. Kapovich-Schupp.

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Fact 3 and a result of P. Neumann imply Fact 2.

Proof of Fact 3

Let $G = \langle x_1, ..., x_g \mid u_1, ..., u_r \rangle$, $g - r \geq 2$.

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So $\bar{H}$ is $\langle s \rangle * K$ where K has $(g - 1)(n - m)$ generators and nr relators. For large enough n, then #generators - #relators of K is ≥ 1. So K maps onto $\mathbb{Z}$, and $\bar{H}$ maps onto F_2. Q.E.D.

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- every finite section is solvable; every nilpotent finite section is Abelian.

Example (Magnus procedure).

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$$\langle a, b \mid aba^{-1} \cdot b^{-1} \cdot aba^{-1} \cdot b^{-1} \cdot a^{-1}b^{-1}a = 1 \rangle.$$

Replace $a^i ba^{-i}$ by b_i. The index i is called *the Magnus a-index* of that letter.

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$$\langle a, b_{-1}, b_0, b_1 \mid b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1, a^{-1} b_0 a = b_{-1}, a^{-1} b_1 a = b_0 \rangle.$$

So we have a new presentation of the same group.

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Note that b_{-1} appears only once in $b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1$.

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So we can replace b_{-1} by $b_1 b_0^{-1} b_1 b_0^{-1}$, remove this generator, and get a new presentation of the same group.

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$\langle a, b_0, b_1 \mid a^{-1}b_0a = b_1b_0^{-1}b_1b_0^{-1}, \qquad a^{-1}b_1a = b_0\rangle$. This is clearly an ascending HNN extension of the free group $\langle b_0, b_1\rangle$.

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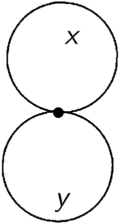

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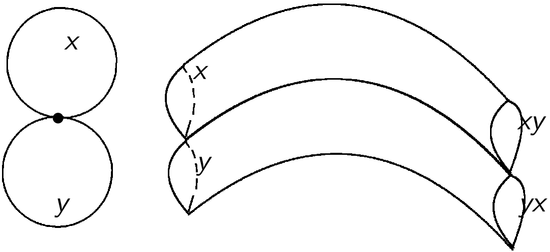

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- (Wise-S.) An ascending HNNextension of a residually finite group can be non-residually finite (example - Grigorcuk's group and its Lysenok extension).

Walks in $\mathbb{Z}^2$

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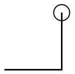

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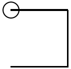

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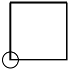

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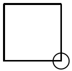

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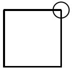

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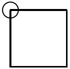

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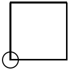

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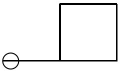

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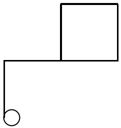

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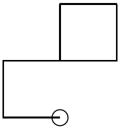

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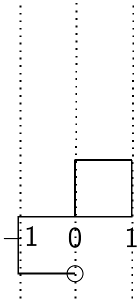

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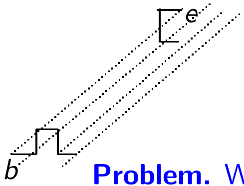

Problem. What is the probability that a support line of the walk intersects the walk only once?

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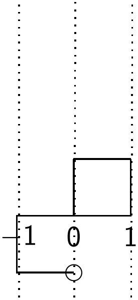

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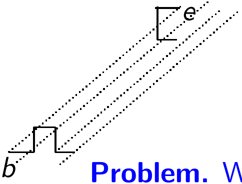

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Dunfield and Thurston proved recently that this probability is strictly between 0 and 1.

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- If $k = 2$ and one of the two support lines of w that is parallel to $\vec{OM}$ intersects w in a single vertex or a single edge, then G is an ascending HNN extension of a free group.
- If $k > 2$ then G is never an ascending HNN extension of a free group.