Rings 1

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You could have noticed that we started a completely new topic, "Rings". I assume that you have an algebra book and can read there or in your notes all the definitions needed to solve these problems.

0 (*) Let R be an (associative unitary) ring. Suppose that $a \in R$ is a nilpotent element. Prove that $1 - a$ is a unit.

1. (*) Let Q be the quaternion ring (see your notes, by the way, it was introduced by Hamilton).

a) For every $q = a + bi + cj + dk \in Q$, let the norm $|q|$ be defined as $\sqrt{a^2 + b^2 + c^2 + d^2}$ (here a, b, c, d are real numbers). Prove that for every $q_1, q_2 \in Q$, we have $|q_1||q_2| = |q_1 q_2|$.

b) Let Q_Z be the subset of Q consisting of all quaternions with integer coefficients. Show that Q_Z is a subring of Q.

c) Show that an element q of Q_Z is a unit of Q_Z if and only if $|q| = 1$.

d) Find all $x \in Q$ satisfying $x^2 = i$?

2. (*) Let R be a ring. The smallest non-zero natural number p such that $p \cdot 1 = 1 + 1 + ... + 1$ (p times) is equal to 0 in R (if such a p exists), is called the *characteristic* of R. If such a p does not exist, we say that R has *characteristic 0*. Prove that the characteristic of a division ring is a prime number or 0.

3. (*) Let G be a group, $H < G$ be a finite subgroup of G, $|H| = n$. Let R be a ring and $R[G]$ be the group ring of G with coefficients from R. Suppose that $n \cdot 1 = 1 + 1 + ... + 1$ (n times) is a unit in R. Prove that the element $e = (n \cdot 1)^{-1} \sum_{h \in H} h$ of $R[G]$ is an idempotent.

4. (*) Show that every abelian group A with operation $+$ becomes an associative (but not unitary) ring if we define the product by $a * b = 0$ for all $a, b \in A$.

5. (*) Let A be an abelian group. Consider the set of all endomorphisms $End(A)$ of A (i.e. all homomorphisms from A to A). If $f, g \in End(A)$ then we define $f + g$ as the map $A \to A$ which takes every a to $f(a) + g(a)$. We also define $f \cdot g$ as the composition of f and g. Show that $End(A)$ is a ring.

6. (*) a) Find all idempotents and nilpotent elements in the ring $\mathbb{Z}/12\mathbb{Z}$.

b) Find all idempotents and nilpotent elements in the ring of 2 by 2 matrices $M_2(\mathbb{Z}/2\mathbb{Z})$.

c) Let F_1, F_2 be division rings. Find all idempotents and nilpotent elements in the direct product $F_1 \times F_2$.

7. Let R be a non-unitary associative ring. Consider the set R^1 consisting of all formal sums $n+r$ where $n \in \mathbb{Z}, r \in R$. Define operations on R^1 by setting $(n+r)+(n'+r') = (n+n')+(r+r')$ and $(n+r)(n'+r') = nn' + (nr' + n'r + rr')$ where nr', for example, is $r' + r' + ... + r'$ (n times). Show that R^1 is a unitary associative ring containing R as a (non-unitary) subring. Thus one can make every ring into a unitary ring.

8. (**) Let $\hat{\mathbb{Z}}_p$ be the ring of all p-adic numbers (see the notes or the book).

a) (***) Prove that if $p = 5$ there exists an element x of $\hat{\mathbb{Z}}_5$ satisfying $x^2 = -1$.

b) (*) Prove that if $p = 3$ then there is no $x \in \hat{\mathbb{Z}}_3$ satisfying $x^2 = -1$.