Math 204, Sample Final Exam

April 27, 2010

This list of sample exam problems. You should not expect the actual exam to be a subset of the problems you see here. Do not assume that the problems given here cover every possible "type" of problem. The final will have about 10 questions on it.

Answers should be **explained and justified**. You may quote results that are either proven in the book or that we have proven in class (unless the question asks you to prove something). You will **lose credit for making false statements**.

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1. Find the range and null space of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -1 & 3 \\ 1 & 3 & -2 & 9 \end{bmatrix}$. Also find a basis for the range and the null space.

2. Show that the inverse of an upper-triangular matrix, when it exists, is upper-triangular.

3. Find all matrices X that commute with each the following matrices

   (a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$,

   (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

4. Consider the perpendicular (orthogonal) projection P that projects vectors in $\mathbb{R}^3$ onto the plane $ax_1 + bx_2 + cx_3 = 0$. You may assume that P is linear.

   (a) Describe the linear transformation $I - P$ geometrically (draw a picture).

   (b) Using the information in the previous part, find the matrix of P.

5. Find a polynomial f of degree 3 such that f passes through the points $(1, 1)$, $(2, -1)$ and $f'(0) = f'(3) = 0$.

6. Decide whether the following sets are subspaces of $\mathbb{R}^4$. Find a basis for the set in the cases where it is a subspace.

   (a) $\{(x_1, x_2, x_3, x_4) : x_1^2 = x_2^2 \text{ and } x_3^2 = x_4^2\}$

   (b) $\{(x_1, x_2, x_3, x_4) : x_1 = 2x_2 \text{ and } x_3 = 2x_1 - 3x_4\}$

7. Decide whether each of the following statements if true or false. A correct answer will receive full credit, a wrong answer carries a half-credit penalty.

   (a) Every $m \times n$ matrix can be brought to row-reduced form by at most n elementary row operations.

   (b) If $m > n$ and $A \in \mathbb{R}^{m\times n}$, then $Ax = b$ can be solved for every choice of vector b.

   (c) If $m = n$, does the system $Ax = b$ have to inconsistent for some choice of vector b.

   (d) If $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ with $AB = I$, then there is a matrix $C \in \mathbb{R}^{m\times n}$ such that $CA = I_m$.

   (e) If W and V are subspaces, then $W \cap V$ is a subspace.

   (f) The span of m vectors in $\mathbb{R}^n$ is an m-dimensional subspace.

8. Find an orthonormal basis for the range of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$. Also find the matrix of the projection onto the kernel of A^T.

9. (a) If A is a symmetric matrix, then show that range(A) and ker(A) are perpendicular.

   (b) If A is an $m \times n$ matrix, then show that the projection onto the range of A is given by $A(A^TA)^{-1}A^T$.

10. Suppose that P is the orthogonal projection onto an m-dimensional subspace of $\mathbb{R}^n$. Explain why there is an orthonormal basis of $\mathbb{R}^n$ in which the matrix of P is of the form

$$\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}.$$

11. Show that $\mathcal{T}$ the collection of upper-triangular matrices is a vector space with respect to matrix addition and scalar multiplication. Show that $\dim(V) = \dfrac{n(n+1)}{2}$.

12. Consider the function $T : C^\infty \to C^\infty$ given by $T(f) = f'' + f$.

    (a) Is the function T linear?

    (b) Find the kernel of T and compute the nullity of T, i.e., the dimension of the kernel.

    (c) What is the range of T?

13. Suppose that V is a subspace of $\mathbb{R}^n$. Show that $V + V^\perp = \mathbb{R}^n$ and that $V \cap V^\perp = \{\vec{0}\}$.

14. Suppose that V and W are subspaces of $\mathbb{R}^n$. Show that $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$.

15. If P is the orthogonal projection onto a subspace V, then show that $I - P$ is the projection onto $V^\perp$.

16. Suppose that $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$ is an orthonormal basis for a subspace $V \subset \mathbb{R}^n$. Let A be the $n \times m$ matrix with columns $\vec{u}_1, \ldots, \vec{u}_m$. Let $P = \text{proj}_V$ be the the orthogonal projection onto V, i.e., $P = \text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \cdots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m$. Show the following:

(a) $A^T A = I_m$.

(b) $AA^T = P$.

(c) $P^2 = P = P^T$.

(d) $(I - P)^2 = (I - P) = (I - P)^T$.

(e) $\text{range}(I - P)^{\perp} = \text{range}(P)$.

17. Find the least-squares solution for the case where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Is the solution unique? Why or why not.

18. Let $v_1 = e_1, v_2 = e_2 + e_1$ and $v_3 = e_1 + e_2 + e_3$.

(a) Show that $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

(b) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$. Find the matrix of A in the basis $\mathcal{B}$.

19. Suppose that A is an upper-triangular matrix. Is there necessarily a basis in which the matrix of A is symmetric?

20. Suppose that the characteristic polynomial of A is $p_A(t) = (t - 1)^3(t + 2)^4(t - 3)(t + 5)^2$.

(a) What is the size of the matrix A?

(b) How many eigenvalues does A have, and what are the corresponding algebraic multiplicities.

(c) What can you conclude about the geometric multiplicities.

(d) Suppose that I tell you that A is upper-triangular, does this give you more information about the geometric multiplicities? Explain what and why.

(e) Suppose that I tell you that A is symmetric, does this give you more information about the geometric multiplicities? Explain what and why.

21. Suppose that P is an orthogonal projection.

(a) Show that the eigenvalues of P are 0 and 1.

(b) Show that $p_P(t) = t^{n-m}(t - 1)^m$ where $m = \text{rank}(P)$.

22. Consider the matrix $A \in \mathbb{R}^n$ that has every entry equal to 1.

(a) Show that $A^2 = nA$.

(b) Show that the eigenvalues of A are $\{0, n\}$.

(c) What are the algebraic and geometric multiplicities of these eigenvalues?

23. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & -2 \end{bmatrix}$. Find the eigenvalues of A.

24. Suppose that A is a 2×2 such that the entries are non-negative and the sum of the entries in each row is equal to 1. Find the eigenvalues of A and the corresponding eigenvectors. Is the matrix A diagonalizable? Why or why not.