Math 204, Sample problems (solutions)

① $A = \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -1 & 3 \\ 1 & 3 & -2 & 9 \end{bmatrix}$, $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis for the range is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

A basis for the null space is given by $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix} \right\}$

$c_1 - c_2 - c_3 = 0$

$3c_2 - c_4 = 0$

② Let $A = [a_{i,j}]$ where $a_{i,j} = 0$ if $i > j$. Suppose that B is the inverse of A so that $BA = AB = I$.

The (i,j) entry of AB is the number $\sum_{k=1}^{n} a_{ik} b_{kj}$ which $= 1$ if $i = j$, 0 if $i \neq j$.

Consider $\sum_{k=1}^{n} a_{nk} b_{kn} = 1 = a_{nn} b_{nn}$ (because $a_{n,k} = 0$ for $k < n$).

So $a_{nn} \neq 0$.

Now $\sum_{k=1}^{n} a_{n,k} b_{k,j} = 0$ for $j < n$ so we get

So $a_{nn} b_{n,j} = 0$ ($\because$ $a_{n,k} = 0$ for $k < n$)

So $b_{n,j} = 0$ for $j < n$.

Therefore A is of the form $\left[\begin{array}{c|c} A_1 & x \\ \hline 0 & a_{nn} \end{array}\right]$ and B is of the form $\left[\begin{array}{c|c} B_1 & y \\ \hline 0 & b_{nn} \end{array}\right]$ where $A_1, B_1 \in \mathbb{R}^{(n-1)\times(n-1)}$.

$AB = \begin{bmatrix} A_1 B_1 & A_1 y + b_{nn} x \\ 0 & a_{nn} b_{nn} \end{bmatrix} = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & 1 \end{array}\right]$. Hence, $A_1 B_1 = I_{n-1}$ ①

by induction we get B_1 is upper triangular, so

$B = \left[\begin{array}{c|c} B_1 & y \\ \hline 0 & b_{nn} \end{array}\right]$ is upper triangular.

(3) (a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} X = X \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

we get, $\begin{bmatrix} a+2c & b+2d \\ -a+3c & -b+3d \end{bmatrix} = \begin{bmatrix} a-b & 2a+3b \\ c-d & 2c+3d \end{bmatrix}$

Hence,

$a+2c = a-b \Rightarrow -b = 2c \Rightarrow b+2c = 0$

$2a+3b = b+2d \Rightarrow 2a+2b-2d = 0$

$-a+3c = c-d \Rightarrow -a+2c+d = 0$

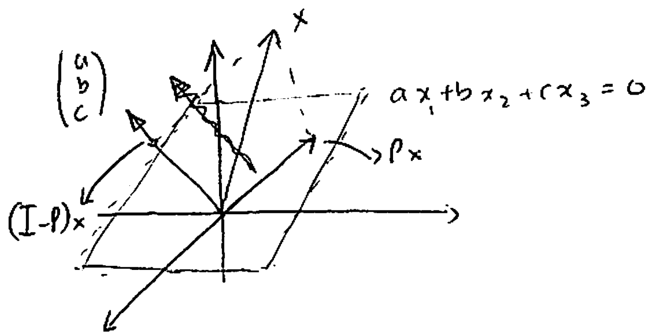
$-b+3d = 2c+3d \Rightarrow -b = 2c$

This is a system $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 2 & 2 & 0 & -2 \\ -1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This matrix has rank 2 and so there is only one vector in the basis of the kernel.

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = [$

④ (a) $I - P$ is the projection onto the line spanned by the vector (a, b, c).



(b) The matrix of $I - P$ is the matrix of the projection onto $\text{Span}\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$

an onb of this space is $u = \dfrac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Hence $P = uu^T = \dfrac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

So $I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \dfrac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

⑤ $f(1) = 1, \quad f(2) = -1, \quad f'(0) = f'(3) = 0$

$$f = a + bt + ct^2 + dt^3$$

$$f(1) = a + b + c + d = 1$$

$$f(2) = a + 2b + 4c + 8d = -1$$

$$f'(0) = b = 0 \implies b = 0$$

$$f'(3) = 3b + 6c + 27d = 0$$

Hence
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 8 \\ 0 & 6 & 27 \end{bmatrix} \begin{bmatrix} a \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The solution is given by row reduction.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 4 & 8 & -1 \\ 0 & 6 & 27 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 7 & -2 \\ 0 & 6 & 27 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 7 & -2 \\ 0 & 0 & 13 & 4 \end{array}\right]$$

$$d = \frac{4}{13}, \quad 3c = -2 - 7d = -2 - \frac{7-4}{13} = -\frac{26-28}{13} = -\frac{54}{13}$$

$$\text{so } c = -\frac{18}{13}$$

$$a + c + d = 1 \implies a = 1 - c - d = 1 - \frac{4}{13} + \frac{18}{13} = \frac{13 - 4 + 18}{13} = \frac{27}{13}$$

$$f(t) = \frac{27}{13} - \frac{18}{13}t^2 + \frac{4}{13}t^3.$$

(6)(a) No $(1,-1,0,0)$, $(+1,1,0,0)$ are in the set but the sum $(2,0,0,0)$ is not.

(b) Yes the set is the kernel of $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}$.

(7) (a) F (b) F (c) F (d) F (e) T (f) F

(8) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$. $c_1 + c_2 + c_3 = 0$, so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the kernel

$\quad\;\; c_1 \quad c_2 \quad c_3$ $\{c_1, c_2\}$ form a basis for range(A).

ONB of range(A) $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\tilde{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

kernel(A^T) = range$(A)^\perp$. Now $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \perp$ range(A) if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

$x_2 + x_3 = 0$ $x_1 - x_3 = 0$

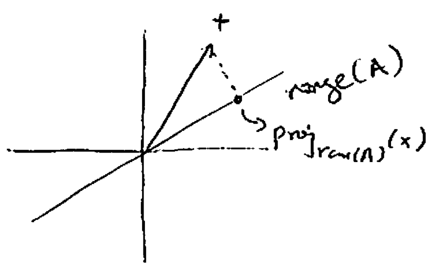
So $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is a basis for kernel(A^T).

Orthonormalize: $u = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$$\text{proj} = \cancel{\tfrac{1}{3}} uu^T = \frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{bmatrix}.$$

9 (a) $\text{range}(A)^{\perp} = \ker(A^T) = \ker(A)$.

(b) Let $x \in \mathbb{R}^n$. $A^T x = A^T A y$ (the least squares equation)



where $Ay = \text{proj}_{\text{ran}(A)}(x)$

Need to get rid of y.

So $y = (A^TA)^{-1}A^T x$

So $Ay = A(A^TA)^{-1}A^T x = \text{proj}_{\text{ran}(A)}(x)$.

10 Let $\{v_1, \dots, v_m\}$ be an orthonormal basis of V and let $\{w_1, \dots, w_k\}$ be an orthonormal basis of $V^{\perp}$.

Since P is the orthogonal projection onto V we get

$Pv_j = v_j$ and $Pw_j = 0$.

So in the basis $\{v_1, \dots, v_m, w_1, \dots, w_k\}$ the matrix of P is of the form

$$\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \overbrace{1 \; \ddots \; 1}^{m \text{ times}} & & \\ & 0 & \\ & & \ddots \; 0 \end{bmatrix}$$

(11) T is upper triangular matrices. We've seen that it's a subspace, see the notes. ~~A~~ Let $E_{i,j}$ = the matrix that is 1 in the ith row jth column and zero in all entries.

The upper-triangular matrix $A = [a_{i,j}]$ can be written as $\sum_{i=1}^{n} \sum_{j=i}^{n} a_{i,j} E_{i,j}$ since $i > j$ implies $a_{i,j} = 0$.

note this

The basis is $\{\underbrace{E_{1,1}, \ldots, E_{1,n}}_{n}, \underbrace{E_{2,2}, \ldots, E_{2,n}}_{n-1}, \ldots, \underbrace{E_{n,n}}_{1}\}$

we have $1 + 2 + \ldots + (n-1) + n = \frac{n(n+1)}{2}$.

(12) (a)

$$T(cf + g) = (cf+g)'' + (cf+g) = cf'' + g'' + cf + g$$

$$= c(f'' + f) + (g'' + g)$$

$$= cT(f) + T(g)$$

Yes

(b) $\text{kernel}(T) = \{f \mid f'' + f = 0\}$

$f'' = f$ if and only if $f'' = -f$

So $f = \cos, \sin$ (4)

$\text{nullity}(T) = 2$ [why are $\cos, \sin$ linearly independent?]

(c) Can you always solve $f'' + f = g$ for any C^∞ function g.

⑫ <u>why?</u> you should have seen this in calculus.

(13) $x \in V \cap V^{\perp} \Rightarrow x \cdot x = 0 \Rightarrow \|x\|^2 = 0$ so $x = 0$.

Now let $\{v_1, \dots, v_m\}$ be an ONB basis for V. Let $x \in \mathbb{R}^n$ and note that $y = x - \left((x \cdot u_1)u_1 + \dots + (x \cdot u_m)u_m\right) \perp u_1, \dots, u_m$.

So $y \perp V$ and $x = \underbrace{y}_{V^{\perp}} + \underbrace{(x \cdot u_1)u_1 + \dots + (x \cdot u_m)u_m}_{V}$

So $\mathbb{R}^n \subseteq V + V^{\perp} \subseteq \mathbb{R}^n$ and therefor $V + V^{\perp} = \mathbb{R}^n$.

(14) Homework problem

~~(15) This follows from (13) and also by the following argument.~~
~~Let $\{u_1, \dots, u_m\}$ be a ONB basis for V and let $\{w_1, \dots, w_k\}$~~
~~be an ONB for $V^{\perp}$~~

(15) Let $x \in V$ and note that $Px = x$, so $(I - P)x = 0$
On the other hand if $x \in V^{\perp}$, then $Px = 0$ so $(I - P)x = x$.
Therefore, $V^{\perp}$ is fixed by $I - P$ and V is sent to 0.
Hence $I - P$ is the projection onto $V^{\perp}$.

16 (a) $A^T A = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} [u_1 | \cdots | u_m] = [u_i \cdot u_j] = I_m$

(b) $AA^T x = A\left(\begin{bmatrix} x \cdot u_1 \\ \vdots \\ x \cdot u_m \end{bmatrix}\right) = (x \cdot u_1) u_1 + \cdots + (x \cdot u_m) u_m$

(c) $P^2 = (AA^T)^2 = AA^T A A^T = A(I_m)A^T = P$

$P^T = (AA^T)^T = A^{TT} A^T = P$

(d) $(I - P)^2 = I + P^2 - 2P = I + P - 2P = I - P$

$(I-P)^T = I^T - P^T = I - P.$

(e)

range $(I-P)^\perp = \ker (I-P)^T = \ker (I-P)$

$x \in \ker(I-P)$ if and only if $(I-P)x = 0$ if and only if $x = Px$

if and only if $x \in$ range (P)

This last part needs some explaining. we want to prove that range $(P) = \{x \mid Px = x\}$

If $x = Px$, then $x \in$ range (P)

if ~~Rex~~ $x = Py$, i.e, $x \in$ range (P), then $Px = P^2 y = Py = x$

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(17) $Ax=b$ is not consistent.

So we look at $A^TAx = A^Tb$

$$A^TA = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} x = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$x_2 = 0,\ x_1 = 1 \qquad x = e_1.$

Note that A^TA is invertible, $\det \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = 8 - 4 = 4 \neq 0$.

So the least squares solution is unique.

(18) (a) Suppose
$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

Substituting we get

$$c_1 (e_1) + c_2 (e_1 + e_2) + c_3 (e_1 + e_2 + e_3) = 0$$

$$(c_1 + c_2 + c_3) e_1 + (c_2 + c_3) e_2 + c_3 e_3 = 0$$

$\{e_1, e_2, e_3\}$ is a basis and so

$$c_1 + c_2 + c_3 = 0$$
$$c_2 + c_3 = 0$$
$$c_3 = 0$$

which gives $c_1 = c_2 = c_3 = 0$.

(19) No $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is upper triangular but not diagonizable

$\because$ eigenvalues of A are 1 with algebraic multiplicity $= 2$.

But $\ker(A - I) = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is 1 dimensional ~~so~~

Therefore geometric multiplicty is smaller.

(20) (a) $n = 3 + 4 + 1 + 2 = 10$

(b) 4 eigenvalues, $\because$ $1, -2, 3, -5$

algebraic multiplicities are $3, 4, 1, 2$

(c) geometric mults are $\leq 3, \leq 4, 1, \leq 2$

that's equal to 1.

(d) No it doesn't, the matrix could have any of the above geometric multiplicities.

(e) ~~Sig~~ Yes. by the spectral theorem the geometric multiplicities equal the algebraic multiplicities.

(21) (a) $A^2 = nA$ ~~so~~ the i,j entry is $R_i \cdot C_j$

$$R_i \cdot C_j = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \underbrace{1 + \dots + 1}_{n \text{ times}} = n.$$

(b) Suppose $Ax = \lambda x$. Note that $A^2 - nA = 0$

so $(A - nI)Ax = (A - nI)\lambda x = \lambda (A - nI)x = \lambda(\lambda - n) = 0$

$\therefore \lambda = 0$ or $\lambda = n$.

(b) $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

<u>Note:</u> $e_1 = v_1$, $e_2 = v_2 - e_1 = v_2 - v_1$

$e_3 = v_3 - e_2 - e_1 = v_3 - (v_2 - v_1) - v_1$

$= v_3 - v_2$.

$Av_1 = Ae_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1 = v_1$ so $[Av_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$Av_2 = A(e_1 + e_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2e_2 = 2(v_2 - v_1) = -2v_1 + 2v_2.$

So $[Av_2]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}_{\mathcal{B}}$

$Av_3 = A(e_1 + e_2 + e_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$

$= 2e_1 + 5e_2 + 3e_3 = 2v_1 + 5(v_2 - v_1) + 3(v_3 - v_2)$

$= -3v_1 + 2v_2 + 3v_3$

So $[Av_3]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$

$[A]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$

(c) A is symmetric so the spectral theorem applies.
Now $\operatorname{rank}(A) = 1$ so $\dim(\ker(A)) = n-1$ so

$\dim(E_0) = n-1$ and $\dim(E_n) = 1$.

algebraic = geometric multiplicites which are
$n-1$ and 0.

(21) (a) $P^2 = P = P^T$ (by problem 16).

$P^2 - P = P(I-P) = 0$.
If $Px = \lambda x$, then $0 = P(I-P)x = P(1-\lambda)x$

$= (1-\lambda)Px = (1-\lambda)\lambda x$

So $\lambda = 0$, $\lambda = 1$

(b) P is the projection onto $\operatorname{range}(P)$. Every vector in the range is fixed, so $Px = x$ for $x \in \operatorname{range}(P)$.
On the other hand, $Px = 0$ for $x \in \operatorname{range}(P)^{\perp} = \ker(P)$.

geometric multiplicity of $1 = m = \dim(\operatorname{range}(P))$
" " of $0 = n-m = \dim(\ker(P))$

So $p_P(t) = (1-t)^m t^{n-m}$.

(23) $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & -2 \end{bmatrix}$

$$\det(A - tI) = \det \begin{bmatrix} 1-t & 0 & 2 \\ 1 & -1-t & 1 \\ 2 & 0 & -2-t \end{bmatrix} = (1-t)\det\begin{bmatrix} -(1+t) & 1 \\ 0 & -(2+t) \end{bmatrix} + 2\det\begin{bmatrix} 1 & -(1+t) \\ 2 & 0 \end{bmatrix}$$

$$= (1-t)(1+t)(2+t) + 2 \cdot 2(1+t)$$

$$= (1+t)\left[(1-t)(2+t) + 4\right] = (1+t)\left[\cancel{6} - t^2 - t + 6\right]$$

$$= -(1+t)(t^2+t-6) = -(1+t)(t+3)(t-2) = 0$$

$t = -1, 2, -3.$

(24) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad a, b, c, d \geq 0$

But, $a + b = 1$, $c + d = 1$ so we can write

$A = \begin{bmatrix} a & 1-a \\ 1-d & d \end{bmatrix}$ $\quad P_A(t) = t^2 - (a+d)t + (ad - (1-a)(1-d))$

$$= t^2 - (a+d)t + (\cancel{ad} - \cancel{ad} + a + d - 1)$$

$$= t^2 - (a+d)t + (a+d-1)$$

t

$$= (t - (a+d-1))(t-1) = 0$$

So $t = 1$, $t = a + d - 1$

~~A is~~ $\ker(A - I) = \ker\begin{bmatrix} a-1 & 1-a \\ 1-d & d-1 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

However, if $a = 1, d = 1$, then $\ker(A - I)$ is 2-dimensional.

If $1 = a+d-1$, then $a+d=2$. $0 \le a, d$ and $a = 1-b \le 1$
$d = 1-c \le 1$

So $a+d=2$ implies $a=1, d=1$.

Therefore, $A = I_2$ $(a=1, d=1)$, is diagonal

or A has 2 distinct eigenvalues and is ~~diagonizable~~ diagonizable.