MATH 204. EXAMPLES OF MATRICES.

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You may have noticed that linear algebra has a lot to do with matrices. In order to prove theorems and find counterexamples you need a healthy supply of examples. Usually you test your understanding of an idea on small matrices, say 2×2 or 3×3. When the matrices are of this size you can get your hands "dirty" and do concrete calculations.

These notes survey the examples we have come across in class. You should not memorize these examples, you should instead experiment with them; add them, multiply them, invert them, and row-reduce them. Don't do this blindly. Watch what you are doing, how does the matrix behave, does it change in a qualitative way?

You can already do a lot in 2 dimensions. The examples we have seen are the rotations, reflections, projections, shears, and scalings. We have also seen how the space is transformed under each of these transformations.

In these notes we'll look at diagonal matrices, triangular matrices, banded matrices, Hankel and Töplitz matrices. We are studying structure rather. While proficiency in computation is useful, computation does not bring insight unless it is done with a purpose.

For each of these classes of matrices you should try to write down examples of matrices in that class, you should also try to examples that are not in that class. You should ask yourself whether the class of matrices is closed under addition, multiplication, and inversion? You should also try making up your own class of matrices and seeing if they have any of these properties.

When we are trying to define a class of matrices we often have an intuitive feeling for what they should look like.

1. Diagonal matrices

A diagonal matrix is one in which there are numbers on the diagonal and all other entries are zero. Here is a picture of a diagonal matrix:

$$\begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ 0 & * & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & * & 0 \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

Date: February 12, 2010.

The stars indicate that there are numbers in those entries. Here is an example of a 3×3 diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & e \end{bmatrix}$. The condition for a matrix to be diagonal is that the entries that are not on the diagonal should be zero.

Definition 1. A matrix $D = [d_{i,j}] \in \mathbb{R}^{n \times n}$ is called a diagonal matrix if and only if $d_{i,j} = 0$ whenever $i \neq j$.

Given a sequence of numbers $a_1, \ldots, a_n$ we can construct a diagonal matrix by setting $d_{i,i} = a_i$ and $d_{i,j} = 0$ for $i \neq j$. Conversely, every diagonal matrix gives rise to the sequence $a_1 = d_{1,1}, \ldots, a_n = d_{n,n}$. Suppose that we have two diagonal matrices A and B with diagonal sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, i.e.,

$$A = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}, B = \begin{bmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_n \end{bmatrix}.$$

First note that the zero matrix and the identity matrix are diagonal matrices. Let us consider the sum of these matrices. If $C = A + B$, then

$$c_{i,j} = a_{i,j} + b_{i,j} = \begin{cases} a_i + b_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Therefore, C the diagonal matrix corresponding to the sequence $c_i = a_i + b_i$. Now we consider the product $C = AB$. By the formula for the matrix product we have $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$. First assume that $i = j$. Consider the sum $c_{i,i} = \sum_{k=1}^{n} a_{i,k} b_{k,i}$. The terms $a_{i,k}$ is zero for $i \neq k$, so the sum reduces to $c_{i,i} = a_{i,i} b_{i,i} = a_i b_i$. Next consider the case $i \neq j$. Once again the sum reduced to $a_{i,i} b_{i,j}$. However, since $i \neq j$, $b_{i,j} = 0$ and so $c_{i,j} = 0$. We have shown that

$$C = \begin{bmatrix} a_1 b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n b_n \end{bmatrix}$$

Finally, let's look at invertibility. Suppose that A is an invertible diagonal matrix. Hence, there is a matrix X such that $AX = I_n$. Note that we can not assume that X is a diagonal matrix. However, we will show that this is the case. Let $X = [x_{i,j}]$. The (i, j) entry of AX is given by $\sum_{k=1}^{n} a_{i,k} x_{k,j}$. The terms $a_{i,k} x_{k,j} = 0$ for $i \neq k$ and so the sum reduces to $a_{i,i} x_{i,j}$. On the other hand the product $AX = I_n$, hence,

$$a_{i,i} x_{i,i} = 1 \text{ if } i = j$$
$$a_{i,i} x_{i,j} = 0 \text{ if } i \neq j$$

The first of these equations shows that $a_{i,i} \neq 0$ for $i = 1, \ldots, n$ and that $x_{i,i} = \dfrac{1}{a_{i,i}}$. If we then divide by $a_{i,i}$ in the second equation we see that $x_{i,j} = 0$ for $i \neq j$. Hence, X is the diagonal matrix with diagonal entries $\dfrac{1}{a_{1,1}}, \ldots, \dfrac{1}{a_{n,n}}$

Let us take a moment to reflect on what we have shown. We have proven that the zero matrix and the identity matrix are diagonal, and that the sum, product and inverse are again diagonal matrices. We have also shown that a diagonal matrix is invertible if and only if the diagonal entries are non-zero.

2. Triangular matrices

Definition 2. A matrix $T = [t_{i,j}] \in \mathbb{R}^{n\times n}$ is called upper-triangular if and only if $a_{i,j} = 0$ for $i > j$. It is called lower-triangular if $t_{i,j} = 0$ whenever $i < j$.

Here is a picture of an upper-triangular matrix

$$T = \begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

Here are some examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & \pi & 5 \\ 0 & 0 & e \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 2 \\ 0 & 0 & e \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}.$$

Here are some lower-triangular matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & -4 & e \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 1 & t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

We will use the term triangular to mean either upper- or lower-triangular. Once again observe that the identity matrix and zero matrix are triangular. In fact, every diagonal matrix is both lower- and upper-triangular. The sum of two upper-triangular matrices is upper-triangular. We leave the verification as an exercise. The product is a more interesting calculation. Suppose that $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are upper-triangular. Let $C = [c_{i,j}] = AB$. We will analyze the sum $\sum_{k=1}^{n} a_{i,k}b_{k,j}$ in much the same way as for diagonal matrix. However, we will need to be more careful. Let $i > j$ and consider $c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$. As long as $i > k$, the terms $a_{i,k} = 0$ and so the sum reduces to $\sum_{k=i}^{n} a_{i,k}b_{k,j}$. In this sum $k \geq i > j$ by assumption. Hence, $b_{k,j} = 0$, since $k > j$. Therefore, $c_{i,j} = 0$. This establishes the fact that C is upper-triangular.

It is a fact that the inverse of an upper-triangular matrix is upper-triangular, and that an upper-triangular matrix is invertible if and only if the diagonal entries are non-zero. This is a homework exercise, and a proof will be included in these notes after you turn in the homework.

3. Hankel matrices

A matrix A is called a Hankel matrix if the entry $a_{i,j}$ depends only on the value of $i + j$. Let us look at the 3×3 matrix $a_{i,j} = i + j - 1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Note that the values are constant on the "reverse diagonals". Here is another example of a Hankel matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}. \tag{1}$$

The zero matrix is an example of a Hankel matrix, but the identity is not! Note that in order to specify a Hankel matrix we only need to specify the values on the reverse diagonals. If h_i is such a sequence, then $a_{i,j} = h_{i+j}$.

Definition 3. A matrix H whose (i, j) entry is of the form h_{i+j-1} for some sequence of values $h_1, \dots, h_{2n+1}$. is called a Hankel matrix.

We have seen in class that the inverse of the matrix in (1) is not a Hankel matrix. The sum of Hankel matrices is again a Hankel matrix.

Exercise 1. Is the product of two Hankel matrices a Hankel matrix?

4. Töplitz matrices

A matrix $A \in \mathbb{R}^{n\times n}$ is called a Töplitz matrix if it is constant along its diagonals. That is there is a sequence $a_{-n}, \dots, a_n$ such that $t_{i,j} = a_{i-j}$. The identity matrix is a Töplitz matrix, as is the zero matrix. Here are some examples of Töplitz matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & a & b \\ e & d & a \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 2 & \pi \end{bmatrix}.$$

In general a Töplitz matrix has the form

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{-1} & a_0 & \ddots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{-n+1} & a_{-n+2} & \cdots & a_0 \end{bmatrix}$$

A matrix is called *circulant* is it a Töplitz matrix wit the additional property that $a_{-j} = a_{n-j}$ for $1 \le j \le n-1$. This definition states that any row of the matrix is obtained from the previous row by cycling the entries to the right.

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \ddots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$

Here is an example of a circulant matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 4 & 1 & 2 & -1 \\ -1 & 4 & 1 & 2 \\ 2 & -1 & 4 & 1 \end{bmatrix}$$

Exercise 2. Is the sum, product and inverse of a Töplitz matrix a Töplitz matrix. What about circulant matrices?

5. Elementary or Elimination matrices

Definition 4. A matrix $E \in \mathbb{R}^{n \times n}$ is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary row transformation.

Recall that there are three elementary row operations:

(1) Exchange R_i and R_j, denoted $R_i \leftrightarrow R_j$.
(2) Replace R_i by $R_i + cR_j$, denoted $R_i \leftarrow R_i + cR_j$.
(3) Multiply R_i by a non-zero scalar, denoted $R_i \leftarrow cR_i$.

Let's describe the action of the matrix E on a vector x. Suppose that $y = Ex$.

(1) $R_i \leftrightarrow R_j$. $y_k = x_k$ for $k \neq i, j$, $y_i = x_j$ and $y_j = x_i$. In words, the values in the ith and jth positions of x have been switched.
(2) $R_i \leftarrow R_i + cR_j$. $y_k = x_k$ for $k \neq i$ and $y_i = x_i + cx_j$.
(3) $R_i \leftarrow cR_i$. $y_k = x_k$ for $k \neq i$ and $y_i = kx_i$.

Exercise 3. Show that each of the above types of elementary matrix is invertible, and that the inverse in each case is an elementary matrix of the same type.

One use of elementary matrices is as a theoretical tool. Suppose that an elementary matrix $E \in \mathbb{R}^{n \times n}$ is given and that it corresponds to certain elementary row operation. If A is an $n \times m$ matrix, then the matrix EA is the matrix obtained from A by applying the elementary operation that corresponds to E. For instance, if E is the elementary matrix that corresponds to $R_i \leftrightarrow R_j$, then EA is the matrix obtained from A by switching the ith and jth rows of A.

Exercise 4. Give a proof of the above claim about the product EA for each of the three types of elementary matrix E.

6. Permutation matrices

A permutation of the numbers $\{1, \dots, n\}$ is one-to-one, onto function $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. We can think of a permutation as a rearrangement of the numbers from 1 through n. A permutation is usually denoted by the sequence $(\sigma(1), \dots, \sigma(n))$. For instance the function that maps $1 \mapsto 3$, $2 \mapsto 1$ and $3 \mapsto 2$ is a permutation of $\{1, 2, 3\}$. We would write is as $(3, 1, 2)$. Sometimes we denote the permutation by listing the numbers from 1 through n and listing the images below.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

The building blocks of all permutations are the transpositions. A *transposition* is a permutation that interchanges precisely two of the numbers. Therefore a transposition is of the form $\tau(i) = j$, $\tau(j) = i$, and $\tau(k) = k$ for $k \neq i, j$.

A permutation matrix is a matrix obtained from the identity by a permutation of the rows. If σ is a permutation, then the permutation matrix U_σ is has rows $e_{\sigma(1)}, \dots, e_{\sigma(n)}$. The permutation matrices that correspond to transpositions are precisely the elementary matrices that arise from row interchanges. Let us examine the behavior of a permutation matrix. If x is a vector in $\mathbb{R}^n$, then $U_\sigma x = \begin{bmatrix} e_{\sigma(1)} \cdot x \\ \vdots \\ e_{\sigma(1)} \cdot x \end{bmatrix} = \begin{bmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{bmatrix}$. In particular, $U_\sigma e_i = e_{\sigma(i)}$.

Exercise 5. Show that the product of two permutation matrices is again a permutation matrix. If σ, θ are two permutations, then describe the permutation that corresponds to $U_\sigma U_\theta$.

7. Banded matrices

Given an $n \times n$ matrix A we define the kth superdiagonal to the entries $a_{i,j}$ such that $j - i = k$, the kth subdiagonal is the set of entries such that $i - j = k$. When $k = 0$ we get the entries such that $i = j$, i.e., the diagonal. Since $|i - j| \leq n - 1$, we only make this definition for $0 \leq k \leq n - 1$.

We could also have defined the kth diagonal of the matrix to be the entries $a_{i,j}$ such that $j - i = k$, for $-n + 1 \leq k \leq n - 1$. In this case negative values of k correspond to the subdiagonal and positive values correspond to the superdiagonal.

A matrix A is called banded if there is an integer w such that $0 \leq w \leq n - 1$ and $a_{i,j} = 0$ for $|j - i| > w$. the extreme cases are when $w = 0$, in which case we get diagonal matrices, and $w = n - 1$, in which case we get all matrices.

Banded matrices play an important part in numerical linear algebra and numerical analysis.

A matrix is called tridiagonal if $w = 1$, i.e., every entry above the first superdiagonal and below the first subdiagonal is zero. Here is an example of such a matrix:

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

Exercise 6. Is the above matrix invertible? In general is the product of tridiagonal matrices tridiagonal? What about banded matrices? Can you figure out when a general 3×3 tridiagonal matrix is invertible? Can you figure out when an $n \times n$ tridiagonal matrix is invertible? [Warning these last two questions are hard]