MATH 204. PROOF BY INDUCTION.

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These notes are an informal introduction to proof by induction. A more careful presentation of these ideas may be given during the course.

To motivate this technique we begin with a simple problem. What is the sum of the first n natural numbers? What we are looking for is a formula for the sum $1+2+\cdots+n$. This number arises for instance when you want to count the number of handshakes that are required at a business meeting where there are n people, or the number of "clinks" at a toast.

Here is an elegant solution due to Gauss. Write the numbers from 1 to n in reverse and add:

$$\begin{array}{ccccc} & 1 & 2 & \cdots & n \\ + & n & (n-1) & \cdots & 1 \\ \hline = & n+1 & n+1 & \cdots & n+1 \end{array}$$

This calculation shows that $2(1+2+\cdots+n) = n(n+1)$ from which we get

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

We may not be as clever as Gauss. A good starting point is to compute the sum for some small values of n and look for a pattern. Here are the first ten values:

$$1, 3, 6, 10, 15, 121, 28, 36, 45, 55. \tag{1}$$

Do you see a pattern?

We now consider a related problem. We would like to compute the sum $1+2^2+3^3+\cdots+n^2$. The trick we used for the previous example does not seem to work. We write down a few terms and look for a pattern. The first ten terms are

$$1, 5, 14, 30, 55, 91, 140, 204, 285, 385.$$

The pattern is not obvious, is it? Let's try something a little harder. Let's try to find a formula for the sum of the first n cubes, i.e.,, $1+2^3+\cdots+n^3$. Here are the first 10 values:

$$1, 9, 36, 100, 225, 441, 784, 1296, 2025, 3025.$$

These numbers are the squares of the numbers in (1). Having recognized the pattern we would now guess that

$$(1+2^3+\cdots+n^3) = (1+2+\cdots+n)^2.$$

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How do we prove such a claim? Writing down a few special cases is not a proof. Is is also not possible to verify every case by hand.

This is where induction comes in. The method of induction is useful for proving statements that involve the natural numbers. The natural numbers, you will recall, is the set $\{1, 2, 3, \ldots\}$. We denote this set by $\mathbb{N}$.

Suppose that we a sequence of statements P_n. For example the statement P_n could be *the sum of the first n cubes is the square of the sum of the first n natural numbers*, or, more succinctly,

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2(n+1)^2}{4}.$$

The method of induction is used to prove such a sequence of statements as follows:

- We first prove the statement P_1. This is called the *base case*.
- We then assume that P_n is true and prove P_{n+1}. This is called the *induction step*.

Theorem 1. *The sum of the cubes of the first n natural numbers is given by*

$$1 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Proof. Let us denote $\frac{n^2(n+1)^2}{4}$ by f_n.

First the base case. The sum of the first natural number is 1. We substitute $n = 1$ into our expression $\frac{n^2(n+1)^2}{4}$ and get $f_1 = \frac{1^2(1+1)^2}{4} = 1$. This proves the base case.

Next we prove the induction step. Assume that $1 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4} = f_n$. We need to prove the corresponding result for $n + 1$. We have,

$$\begin{aligned}
1 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2^3 + \cdots + n^3) + (n+1)^3 \text{ associative} \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \text{ assumption} \\
&= (n+1)^2\left(\frac{n^2}{4} + (n+1)\right) \\
&= (n+1)^2\left(\frac{n^2 + 4n + 4}{4}\right) \\
&= \frac{(n+1)^2(n+2)^2}{4} = f_{n+1}.
\end{aligned}$$

□

Exercise 1. Prove that $1 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Exercise 2. Can you figure out the formula for $1 + 2^4 + \cdots + n^4$?

Exercise 3. Recall the definition of the factorial of a natural number. We define $n!$ to the be the product of all natural numbers that are less than or equal to n, i.e., $n! = n \times (n-1) \times \cdots \times 2 \times 1$. Here are the first 20 values of $n!$:

1	2	6	24	120
720	5040	40320	362880	3628800
39916800	479001600	6227020800	87178291200	1307674368000
20922789888000	355687428096000	6402373705728000	121645100408832000	2432902008176640000

How many zeros are there at the end of $n!$? Can you prove your answer by induction?