Math 204 Linear Algebra, Homework 7

Due: Friday, March 26th, 2010

Name:
You should print these pages out two-sided (if possible) to save paper, money, and trees. **Staple** your sheets together. Write your answers clearly.

1. Section 5.3, Problem 30.

$\|L(\vec{x})\| = \|\vec{x}\|$

(i) $\ker(L) = \{\vec{0}\}$ (if $L(\vec{x}) = 0$, then $\|L(\vec{x})\| = 0$, so $\|\vec{x}\| = 0$, so $\vec{x} = 0$)

(ii – iv) $\text{rank}(L) = m$, since $\text{nullity}(L) = 0$

Hence $m \leq n$, since $\text{im}(L) \subset \mathbb{R}^n$.

(v) Columns are orthonormal. $A = [\vec{u}_1 | \dots | \vec{u}_m]$

<u>Proof</u>: $\vec{u}_i = L(\vec{e}_i)$, $\|\vec{u}_i\| = \|L(\vec{e}_i)\| = \|\vec{e}_i\| = 1$

$\|\vec{u}_i + \vec{u}_j\|^2 = \|\vec{u}_i\|^2 + \|\vec{u}_j\|^2 + 2\,\vec{u}_i \cdot \vec{u}_j = 2 + 2\vec{u}_i \cdot \vec{u}_j$

but $\|\vec{u}_i + \vec{u}_j\|^2 = \|T(\vec{e}_i) + T(\vec{e}_j)\|^2 = \|T(\vec{e}_i + \vec{e}_j)\|^2 = \|\vec{e}_i + \vec{e}_j\|^2 = 2$

So $2\vec{u}_i \cdot \vec{u}_j = 0$.

(vi) $A^T A = [\vec{u}_i \cdot \vec{u}_j] = I_m$

(vii) $A A^T \vec{x} = A \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_m \end{bmatrix} \vec{x} = A \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_m \cdot \vec{x} \end{bmatrix} = [\vec{u}_1 | \dots | \vec{u}_m] \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_m \cdot \vec{x} \end{bmatrix}$

$= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m = \text{proj}_{\text{im}(A)}(\vec{x})$

2. Section 5.3, Problem 34.

$$\begin{bmatrix} a \\ c \\ e \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 = c \quad , \quad \begin{bmatrix} b \\ d \\ f \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = d = 0$$

$$\begin{bmatrix} a \\ 0 \\ e \end{bmatrix} \cdot \begin{bmatrix} b \\ 0 \\ f \end{bmatrix} = 0 \implies ab + ef = 0$$

$$\left\| \begin{pmatrix} a \\ 0 \\ e \end{pmatrix} \right\|^2 = 1 = a^2 + e^2, \quad \left\| \begin{pmatrix} b \\ 0 \\ f \end{pmatrix} \right\|^2 = 1 = b^2 + f^2.$$

Set $a = \sin(\theta)$, $e = \cos(\theta)$.

Now $b^2 + f^2 = 1$ and $b \sin(\theta) + e \cos(\theta) = 0$

Hence $b = \pm \cos(\theta)$ and $f = \mp \sin(\theta)$
$= \pm e$ $\qquad f = \mp a$

$$\begin{bmatrix} \sin & \cos & 0 \\ 0 & 0 & 1 \\ \cos & -\sin & 0 \end{bmatrix}, \quad \begin{bmatrix} \sin & -\cos & 0 \\ 0 & 0 & 1 \\ \cos & \sin & 0 \end{bmatrix}.$$

3. Section 5.3, Problem 46.

$M = QR, \quad Q^T Q = I$

multiply by Q^T to get $Q^T M = Q^T(QR) = R$.

4. **Section 5.4, Problem 8.**

From class $L^+(\vec{y}) = (A^TA)^{-1}A^T\vec{y}$.

(a) Linear since it's given by a matrix multiplicat

(b) $L^+(\vec{y}) = A^{-1}\cancel{(A^T)^{-1}A^T}\vec{y} = A^{-1}\vec{y} \Rightarrow L^+ = L^{-1}$

(c) $L^+(L(\vec{x})) = (A^TA)^{-1}(A^TA)\vec{x} = \vec{x}$.

(d) $L(L^+(\vec{y})) = A(A^TA)^{-1}A^T\vec{y} = \text{proj}_{\text{im}(A)}(\vec{y})$

(e) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^TA = I_2$ so $L^+(\vec{y}) = A^T\vec{y}$.

5. Section 5.4, Problems 10 and 12.

(a) Suppose $A\vec{x} = \vec{b}$ and note that $\cancel{\vec{x}}$

$$\vec{x} = \mathrm{proj}_{\ker(A)}(\vec{x}) + \mathrm{proj}_{\ker(A)^\perp}(\vec{x})$$

Now $$A\vec{x} = A\,\mathrm{proj}_{\ker(A)}(\vec{x}) + A\,\mathrm{proj}_{\ker(A)^\perp}(\vec{x})$$

$$\vec{b} = \vec{0} + A\,\mathrm{proj}_{\ker(A)^\perp}(\vec{x})$$

so choose $\vec{x}_0 = \mathrm{proj}_{\ker(A)^\perp}(\vec{x})$

(b) If $\vec{x}_0, \vec{x}_1 \in \ker(A)^\perp$ and $A\vec{x}_0 = A\vec{x}_1$, then $\vec{x}_0 - \vec{x}_1 \in \ker(A)$. Hence $\vec{x}_0 - \vec{x}_1 \in \ker(A) \cap \ker(A)^\perp$

so $\vec{x}_0 - \vec{x}_1 = \vec{0}$

(c) Note $\vec{x}_1 - \vec{x}_0$ and $\vec{x}_0$ are perpendicular.

Hence, $$\|\vec{x}_0\|^2 + \|\vec{x}_1 - \vec{x}_0\|^2 = \|\vec{x}_0 + (\vec{x}_1 - \vec{x}_0)\|^2 = \|\vec{x}_1\|^2.$$

However $\|\vec{x}_1 - \vec{x}_0\| \neq 0$, so $\|\vec{x}_0\| < \|\vec{x}_1\|$.

(12) $\vec{x}_0$ is $\underbrace{\text{minimal}}_{\text{part (c)}}$, $\underbrace{\text{unique}}_{\text{part (b)}}$ and a $\underbrace{\text{solution in } \ker(A)^\perp}_{\text{part (a).}}$