Math 204 Linear Algebra, Homework 6

Name:
Due: Friday, March 19th, 2010
You should print these pages out two-sided (if possible) to save paper, money, and trees. **Staple your sheets together.** Write your answers clearly.

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1. Section 3.4, Problem 36.

$$T(\vec{x}) = \vec{v}_1 \times \vec{x} + (\vec{v}_1 \cdot \vec{x})\vec{v}_1$$

$$T(\vec{v}_1) = \vec{v}_1 \times \vec{v}_1 + (\vec{v}_1 \cdot \vec{v}_1)\vec{v}_1 = \vec{v}_1$$

$$T(\vec{v}_2) = \vec{v}_1 \times \vec{v}_2 + (\vec{v}_1 \cdot \vec{v}_2)\vec{v}_1 = \vec{v}_3$$

$$T(\vec{v}_3) = \vec{v}_1 \times \vec{v}_3 + (\vec{v}_1 \cdot \vec{v}_3)v_1 = -\vec{v}_2$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Rotation of $90°$ about the $\vec{v}_1$ axis in the direction from $\vec{v}_2$ to $\vec{v}_3$

2. Section 3.4, Problem 70.

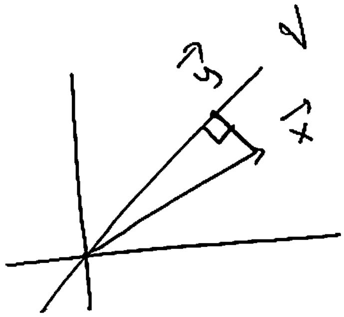
No. If $B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ in the basis $\{\vec{v}_1, \vec{v}_2\}$, then $T\vec{v}_1 = a\vec{v}_1$. But T is a $90°$ rotation and so no vector can be transformed to a multiple of itself.

3. **Section 5.1, Problem 30.**

$\vec{y} = \text{proj}_V(\vec{x})$, so $\vec{y} \perp \vec{x} - \vec{y}$.

Hence $0 = \vec{y} \cdot (\vec{x} - \vec{y}) = \vec{y} \cdot \vec{x} - \|\vec{y}\|^2$

Hence $\|\vec{y}\|^2 = \vec{y} \cdot \vec{x}$ (they're equal!)



4. Section 5.1, Problem 32.

G is invertible if and only if $\det(G) \neq 0$.

Note: $\det(G) = 0$ if and only if G is not invertible

Assume $\det(G) = \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 = 0$

Hence, $|\vec{v}_1 \cdot \vec{v}_2| = \|\vec{v}_1\| \|\vec{v}_2\|$.

This happens if and only if $\vec{v}_2$ and $\vec{v}_1$ are linearly dependent:

Fact: if $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$, then

$$\big\| \|\vec{x}\| \vec{y} - \|\vec{y}\| \vec{x} \big\|^2 = \|\vec{x}\|^2 \|\vec{y}\|^2 + \|\vec{y}\|^2 \|\vec{x}\| - 2 \|\vec{x}\| \|\vec{y}\| \; \vec{x} \cdot \vec{y}$$

$$= 0$$

So $\|\vec{x}\| \vec{y} - \|\vec{y}\| \vec{x} = \vec{0}$, linearly dependent.

5. Section 5.2, Problem 32.

$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a basis for $x_1 + x_2 + x_3 = \vec{0}$

<u>Gram-Schmidt</u>: $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\vec{v}_2 - \mathrm{proj}_{\vec{u}_1}(\vec{v}_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{ONB} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$