Math 204 Linear Algebra, Homework 5

Name:
Due: Friday, March 5th, 2010
You should print these pages out two-sided (if possible) to save paper, money, and trees. **Staple** your sheets together. Write your answers clearly.

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1. Find a basis of $\mathbb{R}^3$ that contains the vector $(1, 2, 3)$. Explain your work.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

Need to show that $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is invertible

$\text{rref}(A) = I_3$ (check this).

2. (a) Show that the collection of vectors in $\mathbb{R}^3$ that satisfy $2x_1 + 3x_2 - 4x_3 = 0$ is a subspace of $\mathbb{R}^3$. Construct a basis for this space.

(b) Generalize your answer to the subspace $ax_1 + bx_2 + cx_3 = 0$.

(a) $V = \left\{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}^{\perp}$

hence V is a subspace

$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4/3 \\ 1 \end{pmatrix} \right\}$

(Check this is a basis!)

(b) if $a = b = c = 0$, then $V = \mathbb{R}^3$

if two of them are zero (say $b = c = 0$) then

$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

if exactly one is zero (say $c = 0$), then

$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -b/a \\ 0 \end{pmatrix} \right\}$.

3. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Express $\vec{v}$ as a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

(b) Check that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$.

(c) The linear transformation T has the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ in the standard basis. Find the matrix of T in the basis of the previous part.

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = S \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix}$, so $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = S^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(check)

(b) Since S is invertible, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis.

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ $B = S^{-1} A S = \begin{bmatrix} 3 & -1 & 3 \\ -4 & 0 & -4 \\ -2 & 0 & -1 \end{bmatrix}$

4. In this exercise you are being asked to give a proof of some of the facts that we mentioned in class. In class we proved the following theorem:

Theorem 1. *Let W be a subspace of $\mathbb{R}^n$. If $\{\vec{w}_1, \ldots, \vec{w}_m\}$ is a spanning set for W and $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly independent set in W, then $p \leq m$.*

Use this theorem to prove the following facts.

(a) Every subspace of $\mathbb{R}^n$ is finite-dimensional.

(b) Any two bases of a subspace W have the same number of vectors. (this is what allows us to define dimension in an unambiguous way).

(c) If V and W are subspaces of $\mathbb{R}^n$, and $V \subseteq W$, then $\dim(V) \leq \dim(W)$.

We begin by proving the following:

Thm: If W is finite dimensional, V is finite-dimensional and $V \subseteq W$, then $\dim(V) \leq \dim(W)$.

Proof induction on $\dim(W)$.

base case: $\dim(W) = 1$. Then $V = \{\vec{0}\}$ or $V = W$ so we're done

Induction: assume true for $\dim(W) = n$ and assume $V \subseteq W$, $\dim(W) = n+1$. If $V = \{\vec{0}\}$, then $\dim(V) = 0 < n+1$

Otherwise choose $\vec{w} \in V$, $\vec{w} \neq 0$

(a) Follows from above by setting $\mathbb{R}^n = W$.

(b) Both bases are lin. indep and spanning

(c) Follows from (a) and above

5. Let V, W be subspaces of $\mathbb{R}^n$. Define $V + W = \{\vec{v} + \vec{w} : \vec{v} \in V, \vec{w} \in W\}$.

(a) Show that $V + W$ is a subspace.

(b) Show that $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$. (If you have taken a probability class this should look very familiar.) Hint: start with a basis of $V \cap W$ and extend it to bases of V and W.

(a) $\vec{0} \in V$, $\vec{0} \in W$, so $\vec{0} = \vec{0} + \vec{0} \in V + W$.

If $\vec{x}, \vec{y} \in V+W$, then $\vec{x} = \vec{v}_1 + \vec{w}_1$ and $\vec{y} = \vec{v}_2 + \vec{w}_2$.

So $c\vec{x} + \vec{y} = \underbrace{(c\vec{v}_1 + \vec{v}_2)}_{\in V} + \underbrace{(c\vec{w}_1 + \vec{w}_2)}_{\in W} \in V + W$

(b) Let $\{u_1, \ldots, u_p\}$ be a basis for $V \cap W$.

Extend this to a basis $\{u_1, \ldots, u_p, v_1, \ldots, v_m\}$ for V

and a basis $\{u_1, \ldots, u_p, w_1, \ldots, w_k\}$ for W.

Claim: $\{\vec{u}_1, \ldots, u_p, w_1, \ldots, w_k, v_1, \ldots, v_m\}$ is a basis for $V+W$. Spanning is straightforward

Lin indep: If $\underbrace{c_1\vec{u}_1 + \cdots + c_p\vec{u}_p}_{\vec{u}} + \underbrace{d_1\vec{v}_1 + \cdots + d_m\vec{v}_m}_{\vec{v}} + \underbrace{b_1\vec{w}_1 + \cdots + b_k\vec{w}_k}_{\vec{w}} = \vec{0}$

then

$\vec{u} + \vec{v} = -\vec{w} \in V \cap W$. Hence, $\vec{w} = \vec{0}$ (note $\vec{w}$ is not in $V \cap W$). So $\vec{w} = 0 = \vec{u} + \vec{v}$. Since $\{w_1, \ldots, w_k\}$ lin indep we get $b_1 = \cdots = b_k = 0$.

Since $\{u_1, \ldots, u_p, v_1, \ldots, v_m\}$ lin indep $c_1 = \cdots = c_p = d_1 = \cdots = d_m = 0$

6. **Bonus problem** Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a function that preserves inner products, i.e., $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Does it follow that T is necessarily linear? What if T preserves distances?