Math 204 Linear Algebra, Homework 5

Name:
Due: Friday, March 5th, 2010
You should print these pages out two-sided (if possible) to save paper, money, and trees. **Staple** your sheets together. Write your answers clearly.

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1. Find a basis of $\mathbb{R}^3$ that contains the vector $(1, 2, 3)$. Explain your work.

2. (a) Show that the collection of vectors in $\mathbb{R}^3$ that satisfy $2x_1 + 3x_2 - 4x_3 = 0$ is a subspace of $\mathbb{R}^3$. Construct a basis for this space.

   (b) Generalize your answer to the subspace $ax_1 + bx_2 + cx_3 = 0$.

3. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

   (a) Express $\vec{v}$ as a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

   (b) Check that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$.

   (c) The linear transformation T has the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ in the standard basis. Find the matrix of T in the basis of the previous part.

4. In this exercise you are being asked to give a proof of some of the facts that we mentioned in class. In class we proved the following theorem:

Theorem 1. *Let W be a subspace of $\mathbb{R}^n$. If $\{\vec{w}_1, \ldots, \vec{w}_m\}$ is a spanning set for W and $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly independent set in W, then $p \leq m$.*

Use this theorem to prove the following facts.

(a) Every subspace of $\mathbb{R}^n$ is finite-dimensional.

(b) Any two bases of a subspace W have the same number of vectors. (this is what allows us to define dimension in an unambiguous way).

(c) If V and W are subspaces of $\mathbb{R}^n$, and $V \subseteq W$, then $\dim(V) \leq \dim(W)$.

5. Let V, W be subspaces of $\mathbb{R}^n$. Define $V + W = \{\vec{v} + \vec{w} \, : \, \vec{v} \in V, \vec{w} \in W\}$.

   (a) Show that $V + W$ is a subspace.

   (b) Show that $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$. (If you have taken a probability class this should look very familiar.) Hint: start with a basis of $V \cap W$ and extend it to bases of V and W.

6. **Bonus problem** Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a function that preserves inner products, i.e., $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Does it follow that T is necessarily linear? What if T preserves distances?