MATH 204 LINEAR ALGEBRA, HOMEWORK 4

Name:

Due: Friday, February 19th, 2010

You should print these pages out two-sided (if possible) to save paper, money, and trees. Staple your sheets together. Write your answers clearly.

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(1) Let A be an $m \times p$ matrix, and let B be an $p \times n$ matrix. Show that the range of A contains the range of AB. Show that the kernel of B is contained in the kernel of AB. Is the reverse inclusion true in either case.

(i) range $(A) \supset$ range (AB)

Suppose that $\vec{x} \in$ range (AB), then there is a vector $\vec{y}$ such that $\vec{x} = AB\vec{y} = A(B\vec{y})$. The vector $A(B\vec{y})$ is in range (A), so $\vec{x} \in$ range (A).

(ii) ker $(B) \subset$ ker (AB)

Let $\vec{x} \in$ ker (B) $\quad B\vec{x} = \vec{0}$, multiplying by A we get $(AB)(\vec{x}) = A(B\vec{x}) = A(\vec{0}) = \vec{0}$. Hence, $\vec{x} \in$ ker (AB).

Reverse inclusions are false:

(i) Let $A = I_n$, $B = 0$, so range $(A) = \mathbb{R}^n$,

range $(AB) = \{\vec{0}\}$

(ii) $B = I_n$, $A = 0$. ker $(B) = \{\vec{0}\}$ and ker $(AB) = \mathbb{R}^n$.

(2) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 4 & 1 \\ 3 & 0 & 1 & -1 \\ 4 & 1 & 13 & 4 \end{bmatrix}$. Find a basis for the range and a basis for the null space of A.

Row reduce A

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 \\ 3 & 0 & 1 & -1 \\ 4 & 1 & 13 & 4 \end{bmatrix} \quad \begin{matrix} \text{III} \leftarrow \text{III} - 4\text{I} \\ \text{II} \leftarrow \text{II} - 3\text{I} \end{matrix} \quad \begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & -3 & -11 & -10 \\ 0 & -3 & -3 & -12 \end{bmatrix} \quad \text{III} \leftarrow \text{III} - \text{II}$$

$$\begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & -3 & -11 & -10 \\ 0 & 0 & 8 & -2 \end{bmatrix}$$

From this we see that $\operatorname{rank}(A) = 3$, so $\operatorname{nullity}(A) = 1$

x_4 is free.

<u>Null space</u>: Solve $A\vec{x} = \vec{0}$ or $\operatorname{rref}(A)\vec{x} = \vec{0}$.

So $8x_3 = 2x_4 \Rightarrow x_3 = \frac{1}{4}x_4$

$$-3x_2 - 11x_3 - 10x_4 = 0 \Rightarrow x_2 = -\frac{11}{3}x_3 - \frac{10}{3}x_4 = -\frac{11}{12}x_4 - \frac{10}{3}x_4$$

$$x_2 = -\frac{54}{\cancel{12}}x_4 = -\frac{9}{2}x_4$$

$$x_1 = -x_2 - 4x_3 - x_4 = -\frac{9}{2}x_4 - x_4 - x_4 = -\frac{13}{2}x_4$$

So $\begin{bmatrix} -13/2 \\ -9/2 \\ 1/4 \\ 1 \end{bmatrix}$ spans $\operatorname{null}(A)$

<u>range (A)</u>

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -11 \\ 8 \end{bmatrix} \right\}.$$

(3) Suppose that A is an $n \times n$ matrix. Show that the equation $Ax = b$ has a unique solution for every choice of b if and only if the only solution to $Ax = 0$ is $x = 0$.

Suppose $A\vec{x} = \vec{0}$ has a unique solution.

If $A\vec{x}_1 = A\vec{x}_2 = \vec{b}$, then $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$ so $\vec{x}_1 - \vec{x}_2 = \vec{0}$ which gives $\vec{x}_1 = \vec{x}_2$.

This shows that there can be <u>at most</u> one solution. Now we need to show there is <u>at least</u> one solution. This follows by noting that if $\mathrm{rref}(A)\vec{x} = \vec{b}$ does not have a solution for some $\vec{b}$, then neither does $A\vec{x} = \vec{b}$. Hence $\mathrm{rref}(A) = I_n$ and so A is invertible

If $A\vec{x} = \vec{b}$ has a unique solution for every choice of $\vec{b}$, then, in particular, this is true for $\vec{b} = \vec{0}$. Hence $A\vec{x} = \vec{0}$ has a unique solution, $\vec{x} = \vec{0}$.

(4) Suppose that W is a subspace of $\mathbb{R}^n$. Define the subspace perpendicular to W by

$$W^{\perp} = \{x \in \mathbb{R}^n : x \cdot w = 0 \text{ for all } w \in W\}.$$

Show that W is a subspace of $\mathbb{R}^n$. If The dimension of W is m, then what is the dimension of $W^{\perp}$? You do not need to prove your answer.

(i) Let $\vec{x}, \vec{y} \in W^{\perp}$, $c \in \mathbb{R}$. Let $\vec{w} \in W$.

We have $(c\vec{x} + \vec{y}) \cdot \vec{w} = c\vec{x} \cdot \vec{w} + \vec{y} \cdot \vec{w}$

$= 0$ (since $\vec{x} \cdot \vec{w} = \vec{y} \cdot \vec{w} = 0$)

Hence, $W^{\perp}$ is closed under linear combinations.

Clearly $\vec{0} \cdot \vec{w} = 0$, so $W \neq \phi$

(ii) $\dim(W^{\perp}) = n - \dim(W)$

(5) Show that the intersection of the subspaces W and $W^\perp$ is the zero subspace.

Suppose $\vec{x} \in W \cap W^\perp$.

then $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x} = 0$, so $\vec{x} = 0$.