Homework 3 Solutions.

2.3 (36) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ This leads to the guess that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

Proof by induction.

When $n = 1$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which agrees with the formula.

Suppose the result is true for n, i.e., that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. We want to prove it for $n+1$.

$$A^{n+1} = A^n A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

(62) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} X = I_2$. Hence, $X \in \mathbb{R}^{3\times 2}$. Say that

$$X = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

We have,

$$\begin{bmatrix} a + 2c + 3e & b + 2d + 3f \\ 4a + 5c + 6e & 4b + 5d + 6f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives two sets of linear equations.

$$a + 2c + 3e = 1 \qquad b + 2d + 3f = 0$$
$$4a + 5c + 6e = 0 \qquad 4b + 5d + 6f = 1$$

Solving these we get.

$$\left|\begin{array}{l} a + 2c + 3e = 1 \\ \quad -3c - 6e = -4 \end{array}\right| \div 3 \rightarrow \left|\begin{array}{l} a + 2c + 3e = 1 \\ \qquad c + 2e = \frac{4}{3} \end{array}\right|$$

So $c = \frac{4}{3} - 2e$, $a = 1 - 2c - 3e = 1 + 4e - \frac{8}{3} - 3e = e - \frac{5}{3}$.

Similarly the second system is

$$\left|\begin{array}{l} b + 2d + 3f = 0 \\ \quad -3d - 6f = 1 \end{array}\right| \rightarrow \left|\begin{array}{l} b + 2d + 3f = 0 \\ 0 + d + 2f = 1/3 \end{array}\right| \, .$$

So, $d = \frac{1}{3} - 2f$, $b = -2d - 3f = -2\left(\frac{1}{3} - 2f\right) - 3f = f - \frac{2}{3}$.

$$X = \begin{bmatrix} e - 5/3 & f - 2/3 \\ \frac{4}{3} - 2e & \frac{1}{3} - 2f \\ e & f \end{bmatrix}$$

<u>Alternative</u> Note that you had to do the same row reduction twice. Instead apply these row operation to both sides.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{3} R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ +\frac{4}{3} & \frac{1}{3} \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} X = \begin{bmatrix} -5/3 & -2/3 \\ 4/3 & 1/3 \end{bmatrix}$$

<u>So now:</u>

$$a - e = -5/3 \qquad b - f = -\frac{2}{3}$$

$$c + 2e = 4/3 \qquad d + 2f = 1/3$$

44 (a) $M_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - 4R_1]{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}} \begin{bmatrix} 1 & 5 & 9 & 13 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{bmatrix}$

$\xrightarrow[R_4 \leftarrow R_4 - 3R_2]{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} 1 & 5 & 9 & 13 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has rank 2.

(b) $M_n = [m_{i,j}]$, $m_{i,j} = (j-1)n + i$

So $R_i = [m_{i,1} \ldots m_{i,n}] = [i, i+n, \ldots, i+n^2-n]$

The first set of row operations is $R_i \leftarrow R_i - iR_1$ for $i \geq 2$

$$R_i - iR_1 = [i, i+n, \ldots, i+n^2-n] - i[1, 1+n, \ldots, n^2-n+1]$$
$$= [0, (1-i)n, \ldots, (1-i)(n^2-n)]$$
$$= (1-i)[0, n, 2n, \ldots, n^2-n]$$

Now the next row operation is $R_i \leftarrow R_i - (i-1)R_2$ for $i \geq 2$

this gives

$$R_i - (i-1)R_1 = (1-i)[0, n, \ldots, n^2-n] - (i-1)[0, -n, \ldots, -n^2+n]$$
$$= \vec{0}$$

Hence, M_n can be reduced to the matrix

$\begin{bmatrix} 1 & n+1 & \cdots & n^2-n+1 \\ 0 & -n & \cdots & -n^2+n \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ which has rank 2.

(c) Since $\text{rank}(M_n) = 2$ and M_n is invertible if and only if $\text{rank}(M_n) = n$, we see that only M_2 is invertible.

(66) Suppose that AB is invertible. So there is a matrix C such that $(AB)C = I_n$. This gives $A(BC) = I_n$. Hence, by the theorem, A is invertible.

Also, $I_n = C(AB) = (CA)B$, which shows that B is invertible.