Dimension of a subspace

Theorem 1. *Suppose that W is a subspace of $\mathbb{R}^n$, that $\{v_1, \ldots, v_p\} \subset W$ is linearly independent, and that $\{w_1, \ldots, w_m\} \subset W$ is a spanning set for W. Then $p \leq m$.*

Proof. The proof is by induction on m. First we prove the base case, where $m = 1$. Since $\{w_1\}$ is a spanning set for W and each of the vectors $v_1, \ldots, v_p \in W$, there exist scalars $c_1, \ldots, c_p$ such that $v_1 = c_1 w_1, \ldots, v_p = c_p w_1$. Note that the numbers $c_1, \ldots, c_p$ are all non-zero, since $v_1, \ldots, v_p$ is a linearly independent set. If $p > m = 1$, then we have

$$c_2 v_1 - c_1 v_2 = c_2(c_1 w_1) - c_1(c_2 w_1) = 0.$$

Hence, $\{v_1, v_2\}$ is linearly dependent, which is a contradiction. Therefore, $p \leq m = 1$.

For the induction step we assume that the theorem is true for m and prove it for the case $m+1$. Suppose that $\{w_1, \ldots, w_{m+1}\}$ is a spanning set for W and that $\{v_1, \ldots, v_p\}$ is linearly independent. There exists scalars $c_{i,j}$, $1 \leq i \leq p$, $1 \leq j \leq m+1$ such that

$$\begin{aligned} v_1 &= c_{1,1} w_1 + \cdots + c_{1,m+1} w_{m+1} \\ &= \\ v_p &= c_{p,1} w_1 + \cdots + c_{p,m+1} w_{m+1} \end{aligned}$$

Since $v_1, \ldots, v_p$ is linearly independent, $v_p \neq 0$. Therefore, at least one of the numbers $c_{p,1}, \ldots, c_{p,m+1}$ is non-zero. We can assume that $c_{p,m+1}$ is non-zero. If this is not the case, then rearrange the vectors $w_1, \ldots, w_{m+1}$. Dividing the last equation by $c_{p,m+1}$ and subtracting we get $w_m = \frac{1}{c_{p,m+1}}(v_p - c_{p,1} w_1 - \cdots - c_{p,m} w_m)$. Substituting for w_m in each of the previous $(p-1)$ equations we get

$$\begin{aligned} v_1 - \frac{c_{1,m+1}}{c_{p,m+1}} v_p &= b_{1,1} w_1 + \cdots + b_{1,m} w_m \\ &= \\ v_{p-1} - \frac{c_{p-1,m+1}}{c_{p,m+1}} v_p &= b_{p-1,1} w_1 + \cdots + b_{p-1,m} w_m \end{aligned}$$

for some scalars $b_{i,j}$. Let $u_i = v_i - \dfrac{c_{i,m+1}}{c_{p,m+1}} v_p$ for $1 \leq i \leq p-1$. Observe that the vectors $u_1, \ldots, u_{p-1}$ are in the subspace spanned by the vectors $w_1, \ldots, w_m$. If we can prove that $u_1, \ldots, u_{p-1}$ are linearly independent, then we can apply the induction hypothesis to the subspace $V = \text{span}\{w_1, \ldots, w_m\}$ and conclude that $p - 1 \leq m$.

We claim that the vectors $u_1, \ldots, u_{p-1}$ are linearly independent. Note that $u_i = v_i + d_i v_p$ for some scalar d_i. Suppose that $c_1 u_1 + \cdots + c_{p-1} u_{p-1} = 0$. Substituting we get,

$$0 = c_1 v_1 + \cdots + c_{p-1} v_{p-1} + (c_1 d_1 + \ldots + c_{p-1} d_{p-1}) v_p.$$

Since $v_1, \ldots, v_p$ is linearly independent, we get $c_1 = \cdots = c_{p-1} = 0$.

By the induction hypothesis we see that $p - 1 \leq m$, or $p \leq m + 1$. This proves the induction step. $\square$