

Name (PRINT clearly):

You will receive full credit only if you show all work and explain all steps.

Disclaimer: This is a practice final exam for Math 170. No claim is being made that it will resemble the actual final exam. In particular, you should not assume that every “type” of problem is listed below.

1. Compute the length of the curve $y = \ln(\sec(x))$ from $x = 0$ to $x = \pi/4$.

Solution: We have $y' = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$. Hence, $\sqrt{1 + (y')^2} = \sqrt{1 + \tan^2(x)} = \sec(x)$. The arc-length is given by

$$\int_0^{\pi/4} \sec(x) dx = \ln |\sec(x) + \tan(x)| \Big|_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln(1) = \ln(\sqrt{2} + 1).$$

2. Find the area of the surface obtained by rotating the graph of $x = \frac{1}{3}(y^2 + 2)^{3/2}$ about the x -axis for $1 \leq y \leq 2$.

Solution: We have $\frac{dx}{dy} = y(y^2 + 2)^{1/2}$. The surface area is

$$\begin{aligned} 2\pi \int_1^2 y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy &= 2\pi \int_1^2 y \sqrt{1 + y^2(y^2 + 2)} dy \\ &= 2\pi \int_1^2 y \sqrt{1 + y^4 + 2y^2} dy \\ &= 2\pi \int_1^2 y(1 + y^2) dy = \frac{1}{2}\pi(1 + y^2)^2 \Big|_1^2 = \frac{21\pi}{2}. \end{aligned}$$

3. The masses m_i are located at the points P_i , for $i = 1, 2, 3$. Find the moments M_x and M_y and locate the centre of mass (\bar{x}, \bar{y}) of the system.

$$m_1 = 6, m_2 = 5, m_3 = 9, P_1 = (1, 5), P_2 = (3, -1), P_3 = (-2, -1).$$

Solution: We have

$$M_x = m_1y_1 + m_2y_2 + m_3y_3 = 6(5) + 5(-1) + 9(-1)$$

$$\bar{x} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{(6)(5) + (5)(-1) + (9)(-1)}{6 + 5 + 9} = \frac{4}{5}.$$

Similarly,

$$\bar{y} = \frac{(6)(1) + (5)(3) + (9)(-2)}{20} = \frac{3}{20}.$$

4. Find the centroid of the region bounded by $y = x^2$, $x = y^2$, $x = 0$ and $x = 1$.

Solution: The area

$$A = \int_0^1 \sqrt{x} - x^2 dx = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

$$\bar{x} = \frac{1}{A} \int_0^1 x(\sqrt{x} - x^2) dx = 3 \left(\frac{2}{5}x^{5/2} - \frac{x^4}{4} \right) \Big|_0^1 = 3 \left(\frac{2}{5} - \frac{1}{4} \right) = \frac{9}{20}.$$

$$\bar{y} = \frac{1}{2A} \int_0^1 (\sqrt{x})^2 - (x^2)^2 dx = 3 \left(\frac{1}{2}x^2 - \frac{x^5}{5} \right) \Big|_0^1 = 3 \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{9}{20}.$$

5. Suppose that the waiting times in a restaurant line are exponentially distributed with a mean waiting time of 4 minutes.

- (a) Write down the formula for the density function.

Solution: The density function is of the form $f(x) = ce^{-cx}$ where c is the reciprocal of the mean. So in this case $f(x) = \frac{1}{4}e^{-x/4}$.

- (b) What is the probability that you will have to wait in line for 4 minutes or longer.

Solution: We have

$$P(X > 4) = \int_4^{\infty} ce^{-cx} dx = \lim_{t \rightarrow \infty} -e^{-cx} \Big|_4^t = \lim_{t \rightarrow \infty} e^{-4c} - e^{-ct} = e^{-4c} = e^{-1} = \frac{1}{e}.$$

- (c) The manager of the restaurant posts the following offer: *If you've been waiting longer than x minutes, then you get a free drink.* The manager does not want to give more than 2% of the customers a free drink. What value of x guarantees this.

Solution: We want the value of x such that $P(X > x) < 0.02$. We have seen from a calculation in the previous part that $P(X > x) = e^{-cx}$. So we need to solve $e^{-cx} < 0.02$ which gives $50 < e^{cx}$. Taking the natural log and dividing we get $x > \frac{1}{c} \ln(50) = 4 \ln(50)$.

6. A chemical reaction can be modelled by the equation

$$\frac{dx}{dt} = k(a-x)(b-x)$$

where t is time and x is the concentration of a certain chemical C .

- (a) Assuming that $a \neq b$ solve the above equation for x in terms of t . Assume that initially C is not present.

Solution: By separation of variables we get

$$\frac{dx}{(a-x)(b-x)} = k dt.$$

partial fractions gives

$$\frac{1}{(a-x)(b-x)} = \frac{1}{b-a} \left(\frac{1}{a-x} - \frac{1}{b-x} \right).$$

Integrating both sides of the equation gives

$$\ln \left(\frac{b-x}{a-x} \right) = k(b-a)t + c.$$

Rearranging and simplifying we get

$$x = \frac{aAe^{k(b-a)t} - b}{Ae^{k(b-a)t} - 1}$$

When $t = 0$, the concentration is 0, i.e., $x = 0$. This gives the equation

$$0 = \frac{aA - b}{A - 1} \Rightarrow A = b/a.$$

Substituting into the original equation we get,

$$x = \frac{a \frac{b}{a} e^{(b-a)kt} - b}{\frac{b}{a} e^{(b-a)kt} - 1} = ab \frac{e^{k(b-a)t} - 1}{be^{(b-a)kt} - a}.$$

- (b) Assuming that $a = b$ solve the equation.

Solution: When $a = b$ we have

$$\frac{dx}{(a-x)^2} = k dt$$

which on integration gives

$$\frac{1}{a-x} = kt + c.$$

This gives $x = a - \frac{1}{kt + c}$.

(c) Describe the behavior of the solution as $t \rightarrow \infty$ in both the above cases.

Solution: We have,

$$\lim_{t \rightarrow \infty} x(t) = \begin{cases} b & \text{if } k(b-a) < 0 \\ a & \text{if } k(b-a) > 0 \end{cases}$$

In the second case we get, $x(t) \rightarrow 0$.

7. Newton's law of gravitation states that the force acting on a body of mass m that is x kilometers above the Earth's surface is given by $F = \frac{mgR^2}{(x+R)^2}$, where m is the mass of the body, g is a constant, and R is the Earth's radius. Remember that force is the mass times acceleration.

(a) Using these facts setup a differential equation to model the velocity $v(x)$.

Solution: We have

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}.$$

We have

$$ma = mv \frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2}.$$

(b) Solve the differential equation and write v as a function of x .

Solution: The equation above is separable and we get

$$v dv = -\frac{gR^2 dx}{(x+R)^2}.$$

Integrating we get

$$\frac{v^2}{2} = \frac{gR^2}{x+R} + C \Rightarrow v = \sqrt{\frac{2gR^2}{x+R} + C}.$$

(c) Describe the behavior of v .

Solution: As $x \rightarrow \infty$, we see that $v \rightarrow \sqrt{C}$.

8. Solve $xy' = y + x^2 \sin(x)$, $y(\pi) = 0$

Solution: We rewrite the equation as

$$y' - \frac{y}{x} = x \sin(x)$$

The integrating factor is

$$I(x) = e^{\int \frac{-1}{x}} = \frac{1}{x}$$

Multiplying the equation by the integrating factor we get

$$\left(\frac{1}{x}y\right)' = \sin(x).$$

Integration gives,

$$\frac{1}{x}y = \int \sin(x) dx = -\cos(x) + C \Rightarrow y = -x \cos(x) + Cx.$$

Use the initial condition $y(\pi) = 0$ to get that $C = -1$. So the solution is $y(x) = -x \cos(x) - x$.

9. Suppose that a population is modeled by a logistic equation. Show that the population is growing fastest when the population is half the carrying capacity.

Solution: The logistic equation is

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right).$$

We compute

$$\frac{d^2P}{dt^2} = \frac{d}{dP} \left(kP \left(1 - \frac{P}{K}\right)\right) \frac{dP}{dt} = \left(k - \frac{2kP}{K}\right) \left(kP \left(1 - \frac{P}{K}\right)\right) = 0.$$

this has solutions $P = 0$, $P = K$ and $P = K/2$. Since the population is assumed positive and never reaches its carrying capacity we see that it grows fastest when $P = K/2$.

10. Decide whether the following sequences converge. If they converge, then compute the limit. Points will be awarded for reasons not guesses.

(a) $a_n = \frac{3^n}{n!};$

Solution: We have,

$$a_n = \frac{3}{n} \times \frac{3}{n-1} \times \cdots \times \frac{3}{1}.$$

For $n \geq 4$ we get that

$$0 \leq a_n \leq \left(\frac{3}{4}\right)^{n-3} \times \frac{9}{2} \rightarrow 0 \text{ since it's geometric.}$$

Applying the sandwich theorem we see that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

(b) $a_n = \ln(n^2 + 1) - 2 \ln(n);$

Solution: We have,

$$a_n = \ln \left(\frac{n^2 + 1}{n^2}\right).$$

We have that $\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^2} = 1$ and so by the continuity of \ln at the point 1 we see that $\lim_{n \rightarrow \infty} a_n = \ln(1) = 0$.

(c) $\arctan(n^2 + 87);$

Solution: The limit of $n^2 + 87$ is infinite. However, $\lim_{x \rightarrow \infty} \arctan(x) = \pi/2$ and so the limit is $\pi/2$.

(d) $a_n = \left(\frac{n}{3+n}\right)^{n/2};$

Solution: We use L'Hopitals rule:

$$\lim_{x \rightarrow \infty} \ln \left(\left(\frac{x}{3+x}\right)^{x/2}\right) = \lim_{x \rightarrow \infty} \frac{x}{2} \ln \left(\frac{x}{3+x}\right) = \lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x}{3+x}\right)}{\frac{2}{x}} = \lim_{x \rightarrow \infty} \frac{-3x^2}{2x(3+x)} = -\frac{3}{2}.$$

Hence, $\lim_{n \rightarrow \infty} a_n = e^{-3/2}$.

(e) $a_{n+1} = 0.98a_n$, for $n \geq 2$, and $a_1 = 1$.

Solution: We have, $a_n = 0.98a_{n-1} = 0.98^2a_{n-2} = \cdots = (0.98)^{n-1}a_1 = (0.98)^n$. This is a geometric series with $r = 0.98 < 1$. Hence, $a_n \rightarrow 0$.

11. Decide whether the following series converge or diverge. Again reasons not guesses.

(a) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!} 2^n$

Solution: Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!2^{n+1}}{(2(n+1))!} \times \frac{(2n)!}{2^n n!} = \frac{2(n+1)}{(2n+1)(2n+2)} = \frac{1}{2n+1} \rightarrow 0 < 1.$$

Converges by the ratio test.

(b) $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n^2+1}}{2n+3} \right)^n$

Solution: Root test:

$$a_n^{1/n} = \frac{\sqrt{n^2+1}}{2n+3} \rightarrow \frac{1}{2} < 1$$

converges.

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n+2}$

Solution: Alternating series. Need to check that $a_n = \frac{\ln(n)}{n+2}$ is decreasing. Let $f(x) = \frac{\ln(x)}{x+2}$. We have,

$$f'(x) = \frac{(x+2)x^{-1} - \ln(x)}{(x+2)^2} = \frac{1 + \frac{2}{x} - \ln(x)}{(x+2)^2}.$$

We need to check the inequality $1 + \frac{2}{x} - \ln(x) < 0$. Note that $\frac{2}{x} \leq 2$, since we consider only $x \geq 1$. Hence, it is enough to check $3 - \ln(x) < 0$. Which is the same as $x > e^3$. Hence, the sequence is ultimately decreasing.

We know that $\lim_{n \rightarrow \infty} a_n = 0$ (use L'Hopital).

By the Alternating series test, the series converges.

(d) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution: We are going to compare the series to the harmonic series. We have,

$$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

Since the limit is non-zero and finite, the comparison test tells us that the series we have behaves like the harmonic series, which diverges.

12. (a) Find the value $1.\overline{316}$ as a ratio of integers.

Solution: We have,

$$1.\overline{316} = 1 + \frac{316}{10^3} + \frac{316}{10^6} + \dots = 1 + \frac{316}{10^3} \sum_{n=0}^{\infty} 10^{-3n} = 1 + \frac{316}{1000} \frac{1}{1 - 10^{-3}} = 1 + \frac{316}{999} = \frac{1315}{999}.$$

- (b) Sum the series $\sum_{n=1}^{\infty} (\ln(n+3) - \ln(n))$

Solution: This is a telescoping series. We have,

$$s_n = \ln(n+3) + \ln(n+2) + \ln(n+1) - \ln(3) - \ln(2) - \ln(1).$$

Since $\ln(n) \rightarrow \infty$ the series above diverges.

13. Consider the following power series $\sum_{n=0}^{\infty} \ln(n+1)x^n$.

- (a) Find the radius of convergence of this power series.

Solution: We have

$$\left| \frac{\ln(n+2)x^{n+1}}{\ln(n+1)x^n} \right| = |x| \frac{\ln(n+2)}{\ln(n+1)} \rightarrow |x|.$$

The reason for this is an application of L'Hopital's rule. We have,

$$\lim_{x \rightarrow \infty} \frac{\ln(x+2)}{\ln(x+1)} = \lim_{x \rightarrow \infty} \frac{x+1}{x+2} = 1.$$

So $R = 1$

(b) Find the interval of convergence.

Solution: At $x = 1$ we get the series $\sum_{n=1}^{\infty} \ln(n+1)$ which diverges since $\ln(n+1) \rightarrow \infty$. For the same reason the series diverges at $x = -1$. So interval = $(-1, 1)$.

(c) Let f be the function that is defined by the above power series on its interval of convergence. Find a power series representation of $f'(x)$. What is the radius and interval of convergence of this power series.

Solution: The series is given by term-by-term differentiation. So we get

$$f'(x) = \sum_{n=1}^{\infty} n \ln(n+1) x^{n-1}.$$

The radius of convergence is unchanged. We have that $n \ln(n) \rightarrow \infty$ so at both endpoints the series diverges. Interval is $(-1, 1)$.

(d) Let F be a function such that $F'(x) = f(x)$ and $F(0) = 1$. Find a power series representation for F , give its radius and interval of convergence.

Solution: Term-by-term integration gives

$$F(x) = c + \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1} x^{n+1}.$$

Since $F(0) = 1$ we get $c = 1$. The radius is unchanged so $R = 1$. At $x = 1$ we have the series $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$. Since $\frac{\ln(n+1)}{n+1} > \frac{1}{n+1}$, the series diverges by a basic comparison to the harmonic series. At $x = -1$ the series is alternating $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n+1)}{n+1}$. This is a lot like the series in problem 11(c) where we saw convergence. Hence interval = $[-1, 1)$.

14. Consider the function $f(x) = e^{-x/2}$

(a) Find an expression for $f^{(n)}(x)$.

Solution: Let's find a few derivatives:

$$f'(x) = -e^{-x/2}/2, f''(x) = e^{-x/2}/4, f'''(x) = -e^{-x/2}/8, \dots, f^{(n)}(x) = (-1)^n e^{-x/2}/2^n$$

(b) Write down an expression for the n th Taylor polynomial for f centered at the origin.

Solution: We have $T_n(x) = 1 - \frac{1}{2}x + \frac{1}{2^2(2!)}x^2 - \dots (-1)^n \frac{1}{2^n(n!)}x^n$.

(c) Estimate the error in using T_n to approximate f on the interval $[-1, 1]$

Solution: The error estimate is $\frac{M|x|^{n+1}}{(n+1)!}$. On the interval we have $|x| \leq 1$. Also $|f^{(n+1)}(x)| = |e^{-x/2}| \frac{1}{2^{n+1}} \leq e^{1/2} \frac{1}{2^{n+1}} \leq \frac{2}{2^{n+1}} = \frac{1}{2^n}$.

Hence, the error is at most $\frac{1}{2^n(n+1)!}$.

(d) Find a reasonable value of n that guarantees that the approximation from the previous part can be carried out with an error no greater than 0.01.

Solution: To approximate we need

$$\frac{1}{2^n(n+1)!} < 0.01 \Leftrightarrow 100 < 2^n(n+1)!$$

This is true for $n \geq 3$, since $2^3(4!) = 8 \times 24 = 192$.