MATH 170, SECTION 01, EXAM 4
Wednesday, December 2nd, 2009

Name (PRINT clearly): (Solutions + key)

- Do not start the test until instructed to do so.
- No calculators.
- You will receive full credit only if you show all work and explain all steps.
- If you run out of space, please ask the instructor for additional sheets of paper. Number the sheets clearly and indicate the problem that is being solved. Scratch work will not be graded.
- The maximum possible score is 150. Point values are indicated on the questions.
- The time limit is 50 minutes.
- There are a total of 3 problems.

Honor pledge

"I pledge on my honor that I have neither given nor received unauthorized aid on this assignment"

Sign here:

1. Find the radius and interval of convergence of the following power series. Justify your answer. (There are three series, one on each page) [50 points]

   (a) $\sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n$

$$\lim_{n\to\infty} \left| \frac{\cancel{(n+1)}(n+2)}{n\,\cancel{(n+1)}} \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} \frac{n+2}{n} |x| = |x|$$

$$\left.\begin{array}{ll} \text{converges if} & |x| < 1 \\ \text{diverges if} & |x| > 1 \end{array}\right\} \quad R = 1$$

Endpoints: $x = 1$ $\quad \sum_{n=1}^{\infty} \frac{n(n+1)}{2}$ diverges because $\frac{n(n+1)}{2} \to \infty \neq 0$

$x = -1$ $\quad \sum_{n=1}^{\infty} \frac{n(n+1)}{2}(-1)^n$ diverges because $(-1)^n \frac{n(n+1)}{2} \not\to 0$

Interval $= (-1, 1)$

(b) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!} x^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \frac{(2n)!}{n!} x \right| = \lim_{n\to\infty} \left| \frac{n+1}{(2n+2)(2n+1)} \right| |x|$$

$$= \lim_{n\to\infty} \frac{1}{2(2n+1)} |x| = 0$$

So this series converges everywhere

$R = \infty$

$I = (-\infty, \infty)$

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}} x^n$

ratio test

$$\lim_{n\to\infty} \sqrt{\frac{(n+1)^2+4}{n^2+4}}\,|x|$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^2+2n+5}{n^2+4}}\,|x| = |x|$$

converges if $|x|<1$
diverges if $|x|>1$ $\Big\}$ $R = \underline{1}$

Endpoints:

$\underline{x=1}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ is divergent compare it to $\sum \frac{1}{n}$

$\underline{x=-1}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

A.S.T

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$$

Also

$$n < n+1 \Rightarrow n^2 < (n+1)^2$$

$$\Rightarrow \sqrt{n^2+4} < \sqrt{(n+1)^2+4}$$

$$\Rightarrow a_{n+1} < a_n.$$

Interval $= [-1, 1)$

2. Find a power series representation (centered at 0) for the following functions. Don't forget to state the radius of convergence and the interval of convergence. [50 points]

(a) $f(x) = \dfrac{x^3}{3+x^2}$

$$\frac{x^3}{3+x^2} = \frac{x^3}{3}\left(\frac{1}{1-\left(\frac{x^2}{3}\right)}\right) = \frac{x^3}{3}\sum_{n=0}^{\infty}\left(\frac{x^2}{3}\right)^n = \cancel{\frac{x^3}{3}}\sum_{n=0}^{\infty}\frac{x^{2n+3}}{3^{n+1}}$$

~~Re~~ converges iff $\left|\frac{x^2}{3}\right| < 1$, i.e, $|x| < \sqrt{3}$

does not converge at end points

$(-\sqrt{3}, \sqrt{3})$, $R = \sqrt{3}$

(b) $f(x) = e^{2x} - e^{-2x}$

$$e^{x} = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad e^{2x} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!} \quad , \quad e^{-2x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$$

$$e^{2x} \cancel{+} e^{-2x} = \cancel{\sigma^2} \sum_{n=0}^{\infty}\frac{(1-(-1)^n)\,x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{1+(-1)^{n+1}}{n!}x^{2n} \quad \Bigg| \quad = \sum_{n=0}^{\infty} 2\frac{x^{4n+2}}{(2n)!}$$

This is fine

$R = \infty$ (because e^x has $R = \infty$)

$I = (-\infty, \infty)$

(c) $f(x) = \ln(5 - 3x)$

$$\ln(5-3x) = -\frac{1}{3}\int \frac{1}{5-3x}\,dx = -\frac{1}{3}\int \frac{1}{5\left(1-\frac{3x}{5}\right)}\,dx$$

$$= -\frac{1}{15}\sum_{n=0}^{\infty}\int\left(\frac{3x}{5}\right)^n = -\frac{1}{15}\sum_{n=0}^{\infty}\int\frac{3^n}{5^n}x^n$$

$$= C + -\frac{1}{15}\sum_{n=0}^{\infty}\frac{3^n}{5^n}\frac{x^{n+1}}{n+1}$$

Set $x=0$, $C = \ln(5)$

$$\ln(5-3x) = \ln(5) - \frac{1}{15}\sum_{n=0}^{\infty}\frac{3^n}{5^n}\frac{x^{n+1}}{n+1}$$

~~Rec~~ geometric converges if $\left|\frac{3x}{5}\right| < 1$, i.e., $|x| < \frac{5}{3}$

$R = 5/3$

$\underline{X = 5/3}$ $\quad -\frac{1}{15}\sum_{n=0}^{\infty}\frac{3^{\cancel{n}}}{5^n}\left(\frac{5}{3}\right)^{n+1}\frac{1}{n+1} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{5}{3}\frac{1}{n+1}$ diverges (harmonic)

$x = -\frac{5}{3}$ $\quad -\frac{1}{15}\sum_{n=0}^{\infty}\frac{(-1)^n 5}{3}\frac{1}{n+1}$ converges by AST.

3. Let $f(x) = x\cos(x)$. Answer the following questions: [50 points]

(a) Find a formula for the first three derivatives of f.

$$f'(x) = \cos(x) - x\sin(x)$$

$$f''(x) = -\sin(x) - x\cos(x) - \sin(x)$$
$$= -2\sin(x) - x\cos(x)$$

$$f'''(x) = -2\cos(x) - \cos(x) + x\sin(x)$$
$$= -3\cos(x) + x\sin(x)$$

(b) Find an expression for the 6th degreee Taylor polynomial (centered at 0) of the function f.

$$\cos(x) \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$x\cos(x) \sim \boxed{x - \frac{x^3}{2!} + \frac{x^5}{5!}} = T_6(x)$$

(c) Estimate the error in using the nth degree Taylor polynomial T_n to approximate f on the interval $[-1, 1]$.

$f^{(n)}(x) \sim n$ trigonometric $+$ x trig.

$|f^{(n)}(x)| \leq n + |x|$ but $|x| \leq 1$

$\leq n+1$

So Error $\leq M \dfrac{|x|^{n+1}}{(n+1)!} \leq \dfrac{(n+2)}{(n+1)!}$

(d) How large a value of n is required for the approximation to be accurate to within 0.005. Give a reasonable value based on your answers to previous parts.

$$\frac{n+2}{(n+1)!} \leq 0.005 = \frac{5}{1000} = \frac{1}{200}$$

$$200 \leq \frac{(n+1)!}{n+2}$$

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720
7	5040

$n+1 = 7$, $n = 6$

$$200 \leq \frac{7!}{8} = \cancel{720}\ \frac{7}{\cancel{8}} \times \cancel{720} \approx 7 \times 90 = 540$$

okay $\boxed{n=6 \text{ works!}}$